# Annuities Certain

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# General terminology

- An annuity is a series of payments made
  * at specified intervals (e.g., yearly - whence the name) called the payment periods
  * for a certain (defined in advance) length of time
- If the length of time is fixed (deterministic), then the annuity is called annuity certain
- If not, then it is called a contingent annuity; an important example is the life annuity
- If the length of time is deterministic and infinite, the annuity is called a perpetuity

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# The definition

- If an annuity is such that the payments are made at the end of the payment periods, then it is called an annuity-immediate
- *Caveat:* The terminology may seem counter-intuitive ...
- If the amount paid at the end of each period is equal to $1, we call this annuity the basic annuity-immediate
- In general, if the payments are equal for all the periods of an annuity, we say that the annuity is level; otherwise, we say it is a nonlevel annuity

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# Notation: Present value

- $a_{\overline{n|}}$ ... denotes the **present value** of all the payments made by a basic annuity-immediate over the length of time equal to $n$ periods
- Assume that the rate of interest is constant and equal to $i$ in each payment period
- Writing the time 0 equation of value (with the help of a time-line), we get

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n = v \cdot \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{i}$$

- If we want to emphasize the exact value of the interest rate used, we write $a_{\overline{n|}i}$

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# Notation: Accumulated value

- $s_{\overline{n}|}$ . . . denotes the **accumulated value at the time of the last payment** of all the payments made by a basic annuity-immediate over the length of time equal to $n$ periods, i.e.,

$$s_{\overline{n}|} = a(n)a_{\overline{n}|}$$

or, equivalently,

$$a_{\overline{n}|} = v(n)s_{\overline{n}|}$$

- In the case of compound interest with the fixed interest rate of $i$ per period, we have that

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

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# Examples

I. Consider an annuity which pays \$500 at the end of each half-year for 20 years with the interest rate of 9% convertible semiannually. Find the present value of this annuity.

⇒

$$500 \cdot a_{\overline{40}|\,0.045} = 500 \cdot 18.4016 = 9200.80$$

II. Roger invests \$1000 at 8% per annum convertible quarterly. How much will he be able to withdraw from this fund at the end of every quarter to use up the fund exactly at the end of 10 years?

⇒ We look at Roger's withdrawals as an annuity-immediate lasting for 40 payment periods. If Roger is to deplete the \$1000 that are presently on his account in exactly 10 years - this precisely means that the present value of the annuity immediate equals \$1000. So, if we denote the level amounts of each withdrawal by $R$, we have the following equation

$$R \cdot a_{\overline{40}|\,0.02} = 1000$$

We get that $R = 36.56$

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# An Example: Loan repayment

- Assume that a \$1000 loan is to be repaid over a 10-year period by level payments at the end of each year. Let the effective rate of interest be 9% per annum. What is the total amount of interest repaid?

⇒ Let $R$ denote the level payment amount. Then the equation of value at the inception of the loan reads as

$$R \cdot a_{\overline{10}|} = 1000$$

Hence, $R = 155.82$

The total amount of interest paid is equal to

$$10 \cdot R - 1000 = 1558.2 - 1000 = 558.2$$

- Caveat: In this example we have ignored the possible rounding off errors that usually result in the last payment being "irregular"; see Algorithm 3.2.14 in the textbook for a more accurate approach

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# An Example: An implementation of Algorithm 3.2.14

- A fund of \$25,000 is to be accumulated by means of deposits of \$1,000 made at the end of every year as long as necessary. Assume that the fund earns an effective interest rate of 8% per annum. Find the number of **regular** (level) deposits that will be necessary, and the size of the final deposit.

# An Example: An implementation of Algorithm 3.2.14 (cont'd)

⇒ The final time $n$ equation of value is

$$1000 \cdot s_{\overline{n|}} = 25000$$

So,

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i} = \frac{(1.08)^n - 1}{0.08} = 25$$

Using your calculator, you get that $n = 15$
After the end of the $14^{th}$ year, i.e., after the first 14 payments were made, the balance in the fund is

$$1000 s_{\overline{14|}} = 24,215$$

With **interest only**, the amount in the fund grows to be

$$24,215 \cdot 1.08 = 26,152$$

We get a somewhat "irregular" answer, that the amount in the fund is greater than the needed amount. Hence, the final payment should be **negative**; namely, $-\$1152$

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## A useful equality

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\begin{aligned}
i + \frac{1}{s_{\overline{n|}\,i}} &= i + \frac{i}{(1+i)^n - 1} \\
&= i \cdot \left[ 1 + \frac{1}{(1+i)^n - 1} \right] \\
&= i \cdot \frac{(1+i)^n}{(1+i)^n - 1} \\
&= i \cdot \frac{1}{1 - v^n} = \frac{1}{a_{\overline{n|}\,i}}
\end{aligned}
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- *Question:* Can we interpret this formula?

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# An Example revisited

- Let us consider the loan repayment problem once more; as we have seen, we can set up a repayment schedule as a level annuity-immediate
- This time we propose a different scheme to repay the loan
- A $1000 loan is to be repaid over a 10-year period in the following way:
  * at the end of each year the interest accrued is repaid
  * the entire principal is repaid at the end of 10 years.

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# An Example revisited (still)

⇒ In order to be able to repay the entire principal at the end of the loan term, we deposit equal amounts of money at the end of every year during the loan term.

Thus, we have to deposit exactly $1/s_{\overline{n|}}$ at the end of every year for every borrowed dollar

On the other hand, the accrued interest must also be paid at the end of every year and that amount is $i$ per every dollar borrowed

Altogether, we should pay

$$1000\left(i+\frac{1}{s_{\overline{n|}}}\right)$$

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If you compare this to the level annual payment amount $R$ which was calculated earlier - you will realize that in that case annual payments were

$$R = \frac{1000}{a_{\overline{n}|}}$$

- These two examples together illustrate formula

$$i + \frac{1}{s_{\overline{n}|\,i}} = \frac{1}{a_{\overline{n}|\,i}}$$

- *Assignment:* Examples 3.2.15, 3.2.17
  Problems 3.2.1,2,3,8

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# Annuities Certain

## The definition

- If an annuity is such that the payments are made at the beginning of the payment periods, then it is called an annuity-due
- If the amount paid at the end of each period is equal to \$1, we call this annuity the basic annuity-due
- $\ddot{a}_{\overline{n}|}$ ... denotes the **present value** of all the payments made by a basic annuity-due over the length of time equal to $n$ periods
- Again, if we assume that the rate of interest is constant and equal to $i$ in each payment period
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# The definition

- If an annuity is such that the payments are made at the beginning of the payment periods, then it is called an annuity-due
- If the amount paid at the end of each period is equal to \$1, we call this annuity the basic annuity-due
- $\ddot{a}_{\overline{n}|}$ ... denotes the **present value** of all the payments made by a basic annuity-due over the length of time equal to $n$ periods
- Again, if we assume that the rate of interest is constant and equal to $i$ in each payment period
- Writing the time 0 equation of value (with the help of a time-line), we get

$$\ddot{a}_{\overline{n}|\,i} = \frac{1 - v^n}{d}$$

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# Notation: Accumulated value

- $\ddot{s}_{\overline{n|}}$ ... denotes the **accumulated value at time** $n$ of all the payments made by a basic annuity-due over the length of time equal to $n$ periods, i.e.,

$$\ddot{s}_{\overline{n|}} = a(n)\ddot{a}_{\overline{n|}}$$

or, equivalently,

$$\ddot{a}_{\overline{n|}} = v(n)\ddot{s}_{\overline{n|}}$$

- Again, in the particular case of compound interest with the fixed interest rate of $i$ per period, we write

$$\ddot{s}_{\overline{n|}\,i} = (1+i)^n \cdot \ddot{a}_{\overline{n|}\,i} = \frac{(1+i)^n - 1}{d}$$

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# Connection between annuity-immediate and annuity-due

- The most basic relationship is obtained through discounting:

$$\ddot{a}_{\overline{n|}\,i} = (1+i)a_{\overline{n|}\,i}$$

  and, equivalently,

$$\ddot{s}_{\overline{n|}\,i} = (1+i)s_{\overline{n|}\,i}$$

- The other basic relationship is obtained by “shifting the time” by one

$$\ddot{a}_{\overline{n|}} = 1 + a_{\overline{n-1|}}$$

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$$\ddot{s}_{\overline{n|}} + 1 = s_{\overline{n+1|}}$$

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# An Example

- Roger wants to accumulate (at least) \$1000 in a fund at the end of 12 years. To accomplish this he plans to make deposits at the end of each year, the final payment being made one year prior to the end of the investment period. How large (at least) should each deposit be if the fund earns 7% effective per annum?

⇒ We are interested in the accumulated value of the investments one year after the last payment, so the equation of value should involve an annuity-due.

Denote the unknown level annual payment by $R$. Then, we can formalize the conditions in the example as

$$R \cdot \ddot{s}_{\overline{11}|} = 1000$$

...and so

$$R = \frac{1000}{1.07 \cdot 15.7836} = 59.21$$

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