More on annuities with payments in arithmetic progression and yield rates for annuities

1. Annuities-due with payments in arithmetic progression

2. Yield rate examples involving annuities

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The Set-up

- n ... the number of time periods for the annuity-due
- P ... the value of the first payment
- Q ... the amount by which the payment per period **increases**
- So, the payment at the beginning of the j^{th} period is

$$P + Q(j-1)$$

- $(I_{P,Q}\,\ddot{a})_{\overline{n}|\,i}$ ... the present value of the annuity described above
- $(I_{P,Q}\,\ddot{s})_{\overline{n}|\,i}$ ... the accumulated value one period after the last payment, i.e., at the end of the n^{th} period

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Formulas for the accumulated and present values

- Recalling the formula for the accumulated value of the corresponding annuity-immediate and discounting by one time-period, we get

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\begin{aligned}
(I_{P,Q}\,\ddot{s})_{\overline{n|}\,i} &= (1+i)\cdot(I_{P,Q}\,s)_{\overline{n|}\,i}\\
&= (1+i)\cdot\left(P\cdot s_{\overline{n|}\,i}+\frac{Q}{i}\cdot(s_{\overline{n|}\,i}-n)\right)\\
&= P\cdot\ddot{s}_{\overline{n|}\,i}+\frac{Q}{d}\cdot(s_{\overline{n|}\,i}-n)
\end{aligned}
$$

- Multiplying throughout by v^n, we obtain

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(I_{P,Q}\,\ddot{a})_{\overline{n|}\,i} = P\cdot\ddot{a}_{\overline{n|}\,i}+\frac{Q}{d}\cdot(a_{\overline{n|}\,i}-n\cdot v^n)
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- In particular, if $P=Q=1$, the notation and the equations can be simplified to

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Formulas for the accumulated and present values: Decreasing payments

- In particular, if $P = n$ and $Q = -1$, then we modify the notation analogously to what was done for annuities-immediate and get

$$(D\ddot{s})_{\overline{n}|\,i} = \frac{n(1+i)^n - s_{\overline{n}|\,i}}{d}$$
$$(D\ddot{a})_{\overline{n}|\,i} = \frac{n - a_{\overline{n}|\,i}}{d}$$

- Note that

$$(Ia)_{\overline{n}|\,i} + (Da)_{\overline{n}|\,i} = (n+1)a_{\overline{n}|\,i}$$
$$(I\ddot{a})_{\overline{n}|\,i} + (D\ddot{a})_{\overline{n}|\,i} = (n+1)\ddot{a}_{\overline{n}|\,i}$$

- *Assignment:* Examples 3.9.15, 3.9.18, 3.9.19
  Problems 3.9.5,6,8
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A Basic Example

- Consider a $10-$year annuity-immediate with each payment equal to \$155.82 which costs $\$1,000$ at time zero. Assume that the underlying per period interest rate equals 0.07.
  Find the yield rate of this investment.

$\Rightarrow$ The accumulated value of all payments at the end of 10 years is

$$155.82 \cdot s_{\overline{10|}\,0.07} = 155.82 \cdot 13.8164 = 2,152.88$$

Denote the annual yield rate by j. The yield rate of the above investment must satisfy

$$1000 \cdot (1+j)^{10} = 2152.88$$

So, $j = 0.0797$

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An Example: Reinvestment of Interest I

- Payments of \$1,000 are invested at the beginning of each year for 10 years. The payments earn interest at 0.07 effective interest rate per annum. The interest can, then, be reinvested at 0.05 effective.

(I) Find the amount in the fund at the end of 10 years.

$\Rightarrow$ In general, assume that there are n payment years. If there is a basic principal deposit of a single dollar, then the interest i is accrued at the end of every year. If we reinvest that interest amount in a secondary account at another effective interest rate j, this means that:

1. The amount on the primary account at time k is equal to $k+1$, for every $k \leq 9$; then, at time n, the amount is still equal to n since no new deposits are made;
2. The investment stream on the secondary account can be described as an arithmetically increasing annuity-immediate with payment at time k equal to $i \cdot k$.

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The accumulated value at the end of the n periods is equal to the sum of the accumulated values on both the primary and the secondary account, i.e.,

$$n + i \cdot (Is)_{\overline{n|}\, j}$$

If the principal is K, then the accumulated value at the end of the n years is

$$K \cdot (n + i \cdot (Is)_{\overline{n|}\, j}) = K \left(n + i \cdot \frac{s_{\overline{n+1|}\, j} - (n+1)}{j} \right)$$

In the present example, $K = 1,000$, $n = 10$, $i = 0.07$ and $j = 0.05$. So, the accumulated value is

$$1000 \left(10 + 0.07 \cdot \frac{s_{\overline{11|}\, 0.05} - 11}{0.05} \right) \approx 14,490$$

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An Example: Reinvestment of Interest III

(II) Find the purchase premium an investor should pay to produce a yield rate of 8% effective.

$\Rightarrow$ We can simply calculate the present value of the above accumulated value. That should be the fair price for the above investment.

$$14,490 \cdot 1.08^{-10} = 6,712.$$

- *Assignment:* Do all the examples in Section 3.10 (only straightforward analytic methods and calculator work; you do not need to do "guess-and-check" or Newton's methods - unless you like them . . .);

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