Chapter 12
The Black-Scholes Formula

Question 12.1.

You can use the NORMSDIST() function of Microsoft Excel to calculate the values for $N(d1)$ and $N(d2)$. NORMSDIST(z) returns the standard normal cumulative distribution evaluated at z. Here are the intermediate steps towards the solution:

$D1$	0.3730
$D2$	0.2230
$N(d1)$	0.6454
$N(d2)$	0.5882
$N(-d1)$	0.3546
$N(-d2)$	0.4118

Question 12.2.

N	Call	Put
8	3.464	1.718
9	3.361	1.642
10	3.454	1.711
11	3.348	1.629
12	3.446	1.705

The observed values are slowly converging towards the Black-Scholes values of the example. Please note that the binomial solution oscillates as it approaches the Black-Scholes value.

Question 12.3.

a)

T	Call-Price
1	7.8966
2	15.8837
5	34.6653
10	56.2377
50	98.0959
100	99.9631
500	100.0000

As T approaches infinity, the call approaches the value of the underlying stock price, signifying that over very long time horizons the call option is not distinguishable from the stock.

b) With a constant dividend yield of 0.001 we get:

T	Call-Price
1	7.8542
2	15.7714
5	34.2942
10	55.3733
50	93.2296
100	90.4471
500	60.6531

The owner of the call option is not entitled to receive the dividends paid on the underlying stock during the life of the option. We see that for short-term options, the small dividend yield does not play a large role. However, for the long term options, the continuous lack of the dividend payment hurts the option holder significantly, and the option value is not approaching the value of the underlying.

Question 12.4.

a)

T	Call Price
1	18.6705
2	18.1410
5	15.1037
10	10.1571
50	0.2938
100	0.0034
500	0.0000

The benefit to holding the call option is that we do not have to pay the strike price and that we continue to earn interest on the strike. On the other hand, the owner of the call option foregoes the dividend payments he could receive if he owned the stock. As the interest rate is zero and the dividend yield is positive, the cost of holding the call outweighs the benefits.

b)

T	Call Price
1	18.7281
2	18.2284
5	15.2313
10	10.2878
50	0.3045
100	0.0036
500	0.0000

Although the call option is worth marginally more when we introduce the interest rate of 0.001, it is still not enough to outweigh the cost of not receiving the huge dividend yield.

Question 12.5.

a) $P(95, 90, 0.1, 0.015, 0.5, 0.035) = 1.0483$

b) $C(1/95, 1/90, 0.1, 0.035, 0.5, 0.015) = 0.000122604$

c) The relation is easiest to see when we look at terminal payoffs. Denote the exchange rate at time t as $X_t = \frac{Y}{E}$.

Then the call option in b) pays (in Euro): $C = \max\left(\frac{1}{X_T} - \frac{1}{90\frac{Y}{E}}, 0\right)$. Let us convert this into yen:

$$C(\text{ in Yen}) = X_T \times \max\left(\frac{1}{X_T} - \frac{1}{90\frac{Y}{E}}, 0\right)$$

$$= \max\left(1 - \frac{X_T}{90\frac{Y}{E}}, 0\right) = \max\left(\frac{90 - X_T}{90\frac{Y}{E}}, 0\right) = \frac{1}{90\frac{Y}{E}} \times \max(90 - X_T, 0)$$

Therefore, the relationship between a) and b) at any time t should be:

$$P(95, \ldots) = X_t * 90 * C(1/95, \ldots).$$

Indeed, we have: $X_t * 90 * C(1/95, \ldots) = 0.000122604 * 95 * 90 = 1.0483 = P(95, \ldots)$

We conclude that a yen-denominated euro put has a one to one relation with a euro-denominated yen call.

Question 12.6.

a) Using the Black-Scholes formula, we find a call-price of $16.33.

b) We determine the one year forward price to be:

$$F_{0,T}(S) = S * \exp(r * T) = \$100 * \exp(0.06 * 1) = \$106.1837$$

c) As the textbook suggests, we need to set the dividend yield equal to the risk-free rate when using the Black-Scholes formula. Thus:

$$C(106.1837, 105, 0.4, 0.06, 1, \mathbf{0.06}) = \$16.33$$

This exercise shows the general result that a European futures option has the same value as the European stock option provided the futures contract has the same expiration as the stock option.

Question 12.7.

a) $C(100, 95, 0.3, 0.08, 0.75, 0.03) = \$14.3863$

b) $S(\text{new}) = 100 * \exp(-0.03 * 0.75) = \$97.7751$
$K(\text{new}) = 95 * \exp(-0.08 * 0.75) = \$89.4676$
$C(97.7751, 89.4676, 0.3, 0, 0.75, 0) = \$14.3863$

This is a direct application of equation (12.5) of the main text. As the dividend yield enters the formula only to discount the stock price, we can take care of it by adapting the stock price before we plug it into the Black-Scholes formula. Similarly, the interest rate is only used to discount the strike price, which we did when we calculated $K(\text{new})$. Therefore, we can calculate the Black-Scholes call price by using $S(\text{new})$ and $K(\text{new})$ and by setting the interest rate and the dividend yield to zero.

Question 12.8.

a) We have to be careful here: Now we have to take into account the dividend yield when calculating the 9-month forward price:

$$F_{0,T}(S) = S * \exp((r - \text{delta}) * T) = \$100 * \exp((0.08 - 0.03) * 0.75) = \$103.8212.$$

b) Having found the correct forward price, we can use equation (12.7) to price the call option on the futures contract: $C(103.8212, 95, 0.3, 0.08, 0.75, \mathbf{0.08}) = \$14.3863$

c) The price we found in part b) and the prices of the previous question are identical. 12.7a, 12.7b and 12.8b are all based on the same Black-Scholes formula, only the way in which we input the variables differs.

Question 12.9.

a) To be very exact we would have to discount tomorrow's dividend. However:

$$PV(\text{Div}) = 2 * \exp(-0.08 * 1/360) = 1.9996 = \$2.$$

We can now deduct the cash dividend from the current stock price and enter the new value into the Black-Scholes formula: $S* = 50 - 2 = 48$. Therefore,

$$C(48, 40, 0.3, 0.08, 0.5, 0) = \$10.2581.$$

We can calculate the price of the American call. It is the maximum of the price of the European call or the value of immediate exercise today: C(American) = $\max(S(0) - K$, C(European)) = $\max(50 - 40, 10.2581) = \max(10, 10.2581) = 10.2581$ = C(European).

It is not optimal to exercise the American call option early.

b) Now, $C(58, 40, 0.3, 0.08, 0.5, 0) = 19.6677$.
C(American) = $\max(S(0) - K$, C(European)) = $\max(60 - 40, 19.6677) = \max(20, 19.6677) = 20 >$ C(European).
In this case, it is actually optimal to exercise the American call option, because the value of immediate exercise is higher than the continuation value (as described by the price of the European call option).

c) It is optimal to exercise the American call option today if the cum dividend stock price less the strike price of the option exceeds the Black-Scholes value of the European option. It is important to remember that only dividend paying stocks entail the possibility of early exercise for American call options.

Question 12.10.

Time decay is measured by the greek letter theta. We will show in the following that the statement of the exercise is not always correct.

We assume $S = 50$, sigma = 0.3, $r = 0.08$, delta = 0, $K = 40$, 50 and 60, and $T = 1$ month, 3 months, ..., 13 months.

We can calculate:

$K = 40$

Time to expiration	Theta	Call price	Dollar change	Perc. change
1 month	−0.010	10.271	−0.010	−0.09%
3 months	−0.012	10.939	−0.012	−0.11%
5 months	−0.012	11.678	−0.012	−0.11%
7 months	−0.012	12.409	−0.012	−0.10%
9 months	−0.012	13.115	−0.012	−0.09%
11 months	−0.011	13.792	−0.011	−0.08%
13 months	−0.011	14.443	−0.011	−0.07%

$K = 50$

	Theta	Call price	Dollar change	Perc. change
1 month	−0.034	1.892	−0.034	−1.82%
3 months	−0.022	3.481	−0.022	−0.63%
5 months	−0.018	4.669	−0.018	−0.39%
7 months	−0.016	5.688	−0.016	−0.28%
9 months	−0.015	6.606	−0.015	−0.22%
11 months	−0.014	7.453	−0.014	−0.18%
13 months	−0.013	8.247	−0.013	−0.16%

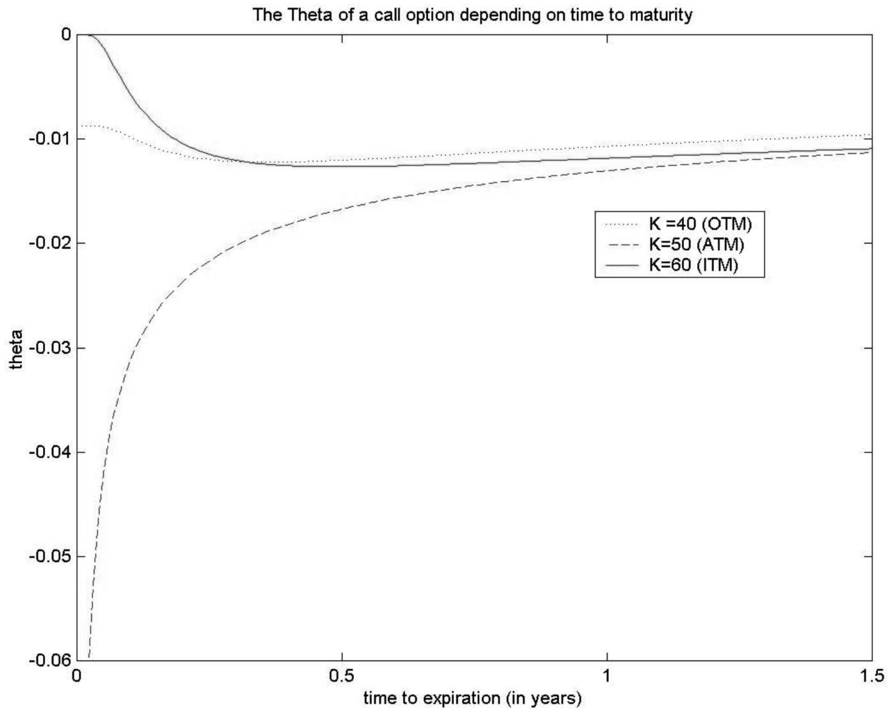
$K = 60$

	Theta	Call price	Dollar change	Perc. change
1 month	−0.004	0.037	−0.004	−11.14%
3 months	−0.012	0.577	−0.012	−2.01%
5 months	−0.013	1.319	−0.013	−0.97%
7 months	−0.013	2.088	−0.013	−0.61%
9 months	−0.012	2.846	−0.012	−0.44%
11 months	−0.012	3.586	−0.012	−0.34%
13 months	−0.012	4.306	−0.012	−0.27%

Please note that we measure theta as the dollar change in the call value per day. Therefore, we divided the returned value of the Excel function BSTheta by 360.

We can see that in fact the statement of the exercise is not correct. Only the at the money call option ($K = 50$) has a monotonically decreasing theta (in time) and thus the greatest time decay for short expirations (i.e., a decreasing dollar and percentage price change if we reduce the time to maturity by one day). Both the out of the money and in the money option have thetas that are not monotonically decreasing in time to maturity, and neither the dollar change nor the percentage change are necessarily greater the shorter the time to expiration is.

In the money and out of the money options can have thetas that are increasing in time to maturity, as the following figure, graphing the theta of the above options, depending on time to maturity, shows:



Question 12.11.

a) Vega is the derivative of the Black-Scholes function with respect to the volatility (sigma). The given formula is approximating this derivative. Epsilon needs to be small because by using the formula we are approximating linearly a non-linear function (recall that a graph of the call vega against the stock price is humpshaped).

b) Assume $S = 100$, $K = 95$, sigma $= 0.3$, $r = 0.08$, delta $= 0.03$ and $T = 0.75$

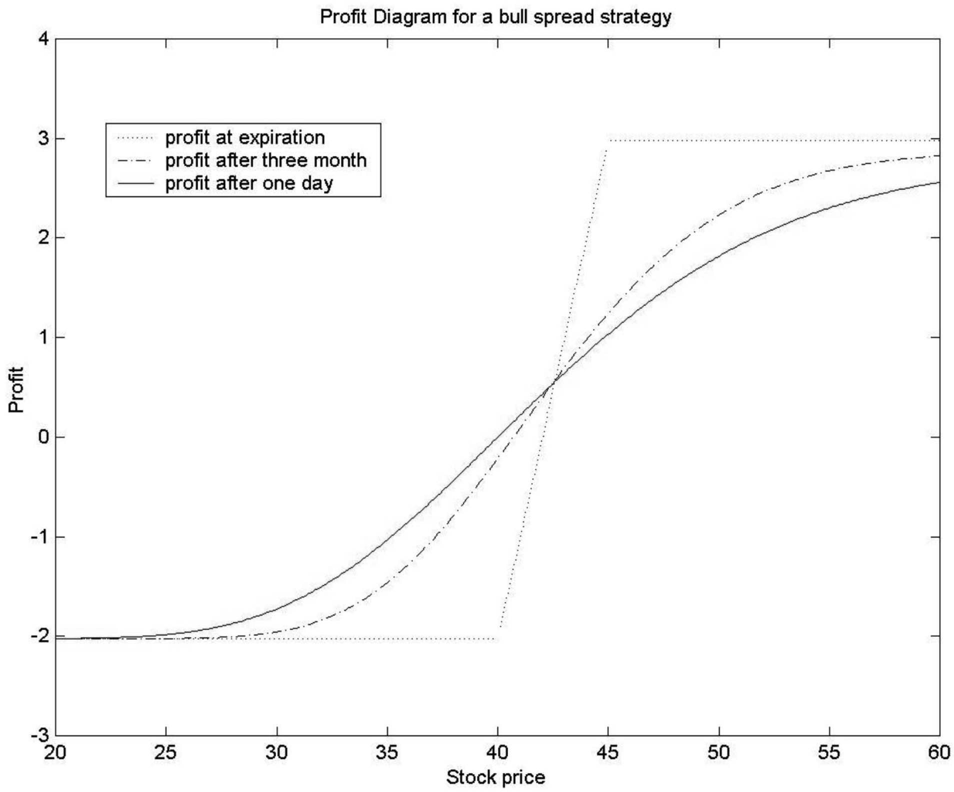
epsilon	call_u	call_d	vega_appr.	BS-vega
0.0001	14.3893	14.3833	0.3022	0.3022
0.0010	14.4165	14.3561	0.3022	0.3022
0.0100	14.6890	14.0846	0.3022	0.3022
0.1000	17.4437	11.4429	0.3000	0.3022
0.2000	20.5318	8.9501	0.2895	0.3022

Question 12.12.

epsilon	call_u	call_d	div_appr.
0.0010	14.3364	14.4363	−0.4997
0.0100	13.8923	14.8917	−0.4997
0.1000	9.9457	19.9526	−0.5003

Let's do a quick check: $C(\ldots, \text{delta} = 0.03) = 14.3863$, $C(\ldots, \text{delta} = 0.04) = 13.8923$. The difference is -0.4940, which is very close to our approximation of -0.4997.

Question 12.13.



Question 12.14.

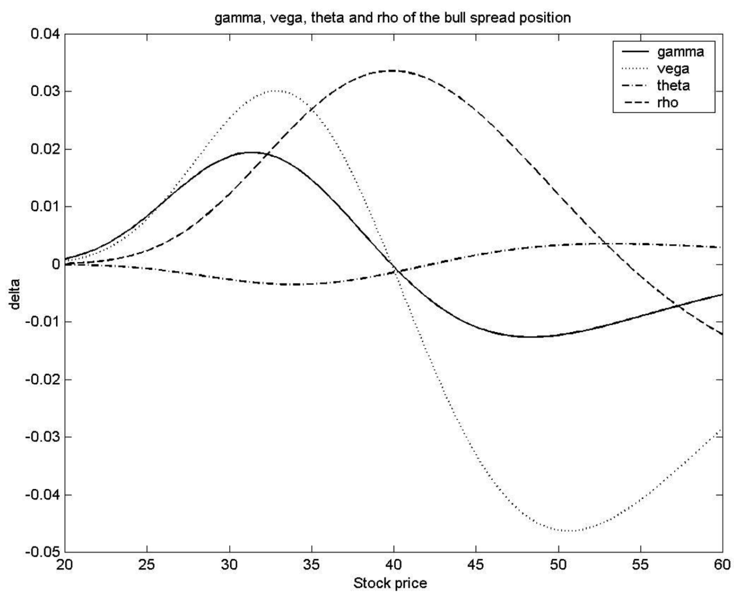
a) The greeks of the bull spread are simply the sum of the greeks of the individual options. The greeks of the call with a strike of 45 enter with a negative sign because this option was sold.

	Bought Call(40)	Sold Call(45)	Bull Spread
Price	4.1553	−2.1304	2.0249
Delta	0.6159	−0.3972	0.2187
Gamma	0.0450	−0.0454	−0.0004
Vega	0.1081	−0.1091	−0.0010
Theta	−0.0136	0.0121	−0.0014
Rho	0.1024	−0.0688	0.0336

b)

	Bought Call(40)	Sold Call(45)	Bull Spread
Price	7.7342	−4.6747	3.0596
Delta	0.8023	−0.6159	0.1864
Gamma	0.0291	−0.0400	−0.0109
Vega	0.0885	−0.0122	−0.0331
Theta	−0.0137	0.0152	0.0016
Rho	0.1418	−0.1152	0.0267

c) Because we simultaneously buy and sell an option, the graphs of gamma, vega and theta have inflection points (see figures below). Therefore, the initial intuition one may have had—that the greeks should be symmetric at $S = \$40$ and $S = \$45$—is not correct.



Question 12.15.

a)

	Bought Put(40)	Sold Put(45)	Bull Spread
Price	2.5868	−5.3659	−2.7791
Delta	−0.3841	0.6028	0.2187
Gamma	0.0450	−0.0454	−0.0004
Vega	0.1080	−0.1091	−0.0010
Theta	−0.0050	0.0025	−0.0025
Rho	−0.0898	0.1474	0.0576

b)

	Bought Put(40)	Sold Put(45)	Bull Spread
Price	1.1658	−2.9102	−1.7444
Delta	−0.1977	0.3841	0.1864
Gamma	0.0291	−0.0400	−0.0109
Vega	0.0885	−0.1216	−0.0331
Theta	−0.0051	0.0056	0.0005
Rho	−0.0503	0.1010	0.0507

c) A similar logic as in exercise 12.14 applies. Because we simultaneously buy and sell an option, the graphs of gamma, vega and theta of the put bull spread also have inflection points.

d) By looking at the formulas in the appendix to chapter 12, we immediately see that the vega and gamma of a put and a call are identical. As we buy and sell the same strikes in exercise 12.14 and 12.15, the vega and gamma of the bull spreads must be the same. The formulas for rho differ for calls and puts (resulting in general in opposite signs), but the payoff structure for the put bull spread and call bull spread have the same shape. Therefore, we may expect a different magnitude, but the same sign and direction.

It is easy to show by put call parity that delta_c −1 = delta_p (for options with the same strike price and time to maturity).

$$\begin{aligned}\text{Now, delta\_bullspread(puts)} &= \text{delta\_p(40)} - \text{delta\_p(45)} \\ &\quad \text{delta\_c(40)} - 1 - (\text{delta\_c(45)} - 1) = \text{delta\_c(40)} - \text{delta\_c(45)} \\ &= \text{delta\_bullspread(calls)}\end{aligned}$$

Therefore, the deltas should be identical.

Question 12.16.

a) 1 day to expiration

S	call delta	put delta	call vega	put vega	call theta	put theta	call rho	put rho
60	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
65	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
70	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
75	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
80	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
85	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
90	0.000	−1.000	0.000	0.000	0.000	0.022	0.000	−0.003
95	0.001	−0.999	0.000	0.000	−0.002	0.021	0.000	−0.003
100	0.509	−0.491	0.021	0.021	−0.326	−0.304	0.001	−0.001
105	0.999	−0.001	0.000	0.000	−0.025	−0.003	0.003	0.000
110	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
115	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
120	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
125	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
130	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
135	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000
140	1.000	0.000	0.000	0.000	−0.022	0.000	0.003	0.000

ab) 1 year to expiration

S	call delta	put delta	call vega	put vega	call theta	put theta	call rho	put rho
60	0.099	−0.901	0.105	0.105	−0.006	0.015	0.052	−0.871
65	0.154	−0.846	0.154	0.154	−0.008	0.012	0.086	−0.837
70	0.220	−0.780	0.207	0.207	−0.012	0.009	0.131	−0.792
75	0.294	−0.706	0.258	0.258	−0.015	0.006	0.184	−0.739
80	0.372	−0.628	0.303	0.303	−0.018	0.002	0.245	−0.678
85	0.450	−0.550	0.336	0.336	−0.021	0.000	0.310	−0.614
90	0.526	−0.474	0.358	0.358	−0.023	−0.003	0.376	−0.547
95	0.597	−0.403	0.368	0.368	−0.025	−0.005	0.442	−0.482
100	0.662	−0.338	0.366	0.366	−0.026	−0.006	0.504	−0.419
105	0.719	−0.281	0.354	0.354	−0.027	−0.007	0.563	−0.360
110	0.769	−0.231	0.335	0.335	−0.028	−0.007	0.617	−0.306
115	0.811	−0.189	0.311	0.311	−0.028	−0.007	0.665	−0.259
120	0.847	−0.153	0.283	0.283	−0.028	−0.007	0.707	−0.216
125	0.877	−0.123	0.254	0.254	−0.027	−0.007	0.743	−0.180
130	0.902	−0.098	0.225	0.225	−0.027	−0.006	0.775	−0.148
135	0.922	−0.078	0.197	0.197	−0.026	−0.006	0.801	−0.122
140	0.938	−0.062	0.171	0.171	−0.025	−0.005	0.824	−0.100

We can clearly see that the entries for the one day expiration table are more extreme: There is only one day left for stock price changes, so a lot of uncertainty is resolved. For example, a deep out of the money call option (e.g. at a stock price of $60) is unlikely to change during one day to some price bigger than $100, so the option most likely does not pay off, therefore its delta is zero. On the other hand, with one year to maturity left, there is a decent chance of such a change, therefore the price of the option reacts to a one dollar increase in the stock price.

ba) Time to expiration: 1 day

S	straddle delta	straddle vega	straddle theta	straddle rho
60	−1.000	0.000	0.022	−0.003
65	−1.000	0.000	0.022	−0.003
70	−1.000	0.000	0.022	−0.003
75	−1.000	0.000	0.022	−0.003
80	−1.000	0.000	0.022	−0.003
85	−1.000	0.000	0.022	−0.003
90	−1.000	0.000	0.022	−0.003
95	−0.999	0.000	0.019	−0.003
100	0.018	0.042	−0.631	0.000
105	0.998	0.000	−0.027	0.003
110	1.000	0.000	−0.022	0.003
115	1.000	0.000	−0.022	0.003
120	1.000	0.000	−0.022	0.003
125	1.000	0.000	−0.022	0.003
130	1.000	0.000	−0.022	0.003
135	1.000	0.000	−0.022	0.003
140	1.000	0.000	−0.022	0.003

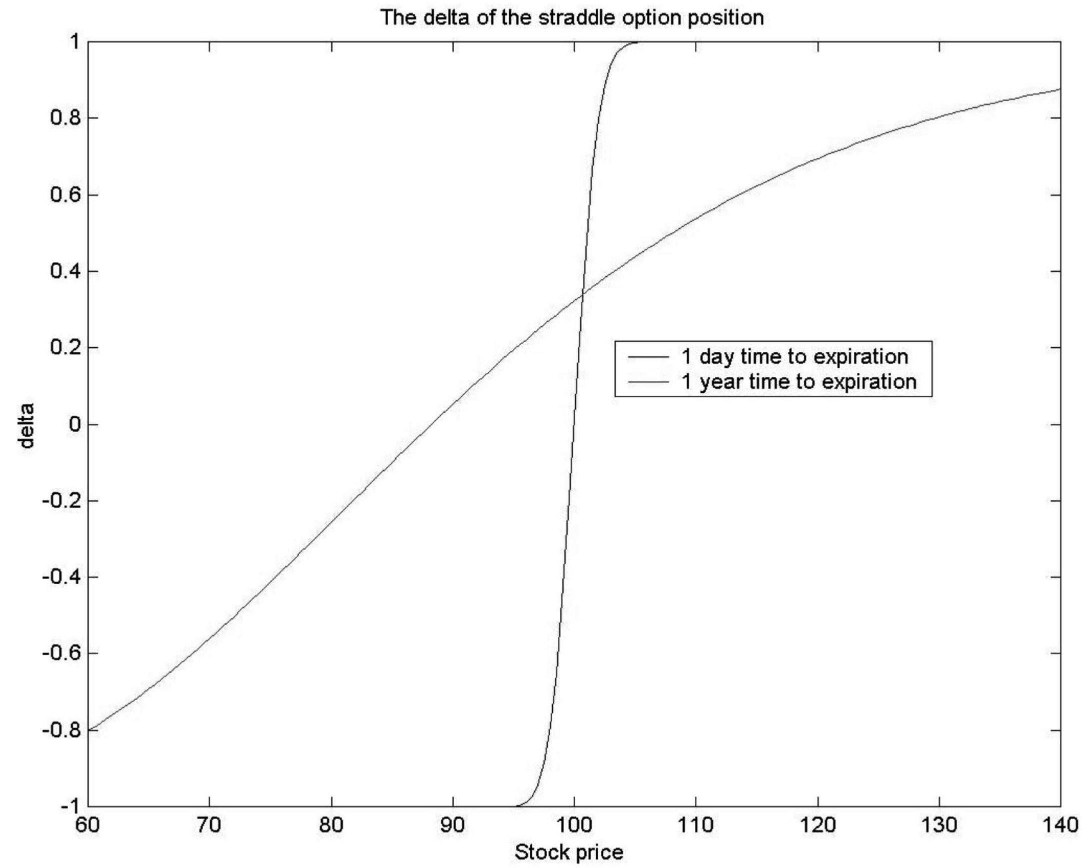
bb) Time to expiration: 1 year

S	straddle delta	straddle vega	straddle theta	straddle rho
60	−0.802	0.209	0.009	−0.819
65	−0.692	0.308	0.004	−0.750
70	−0.560	0.415	−0.003	−0.661
75	−0.412	0.517	−0.009	−0.554
80	−0.256	0.605	−0.016	−0.433
85	−0.100	0.673	−0.021	−0.304
90	0.052	0.717	−0.026	−0.171
95	0.194	0.735	−0.030	−0.040
100	0.323	0.732	−0.032	0.086
105	0.438	0.708	−0.034	0.203
110	0.537	0.670	−0.035	0.310
115	0.623	0.622	−0.035	0.406
120	0.694	0.567	−0.035	0.490
125	0.754	0.509	−0.034	0.564
130	0.803	0.451	−0.033	0.626
135	0.844	0.395	−0.032	0.679
140	0.876	0.342	−0.030	0.724

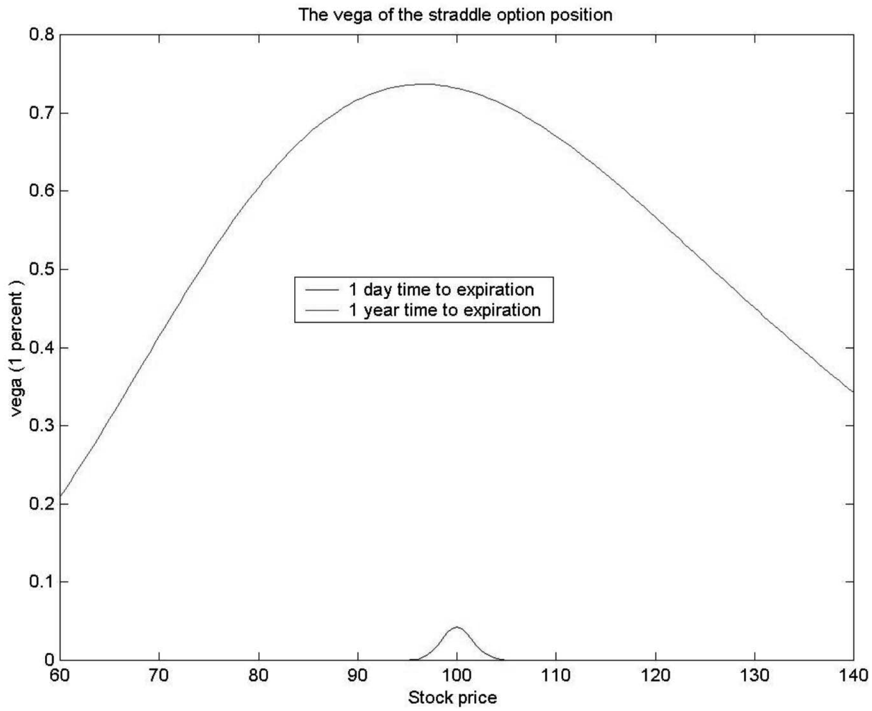
bc) Explanation of the one year greeks

We need to keep in mind that we bought a call option and bought a put option, both with a strike of $100. Therefore, with a stock price smaller than $100, the put option is in the money, and the call option is out of the money. This pattern helps us when we look at the greeks: For small stock prices, delta is negative (the put dominates) and rho is negative (recall that since the put entitles the owner to receive cash, and the present value of this is lower with a higher interest rate, the rho of a put is negative). Deep in the money put options have a positive theta, therefore for very small stock prices, we expect (and see) a positive theta of the straddle. However, once we increase the stock price, the theta of a put becomes negative; the theta becomes progressively more negative as the negative theta effects of the call are integrated. Both put and call have the same vega, and we know that vega is highest for at the money options.

ca)



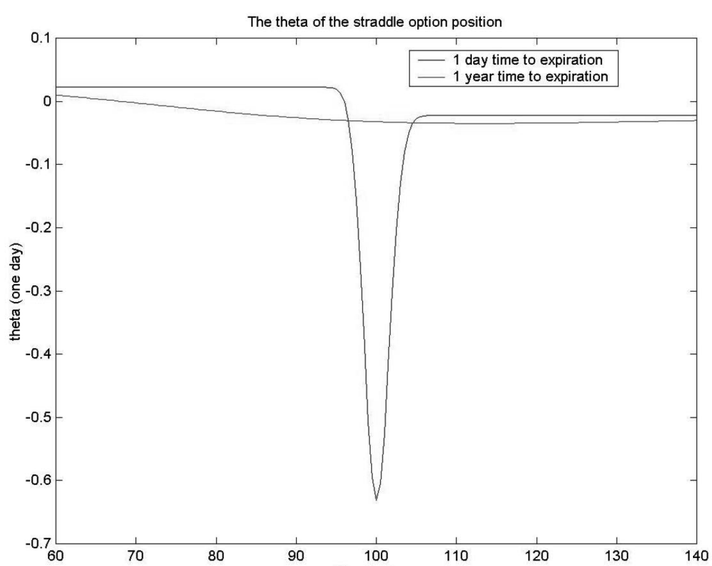
The delta of the one day time to expiration graph is a lot steeper. However, delta changes only in a small area around the strike price. With only one day to expiration left, it becomes increasingly clear whether the call option ends out of the money (delta_c = 0) and the put option ends in the money (delta_p = −1) or the call option in the money (delta_c = 1) and the put option out of the money (delta_p = 0). Taken together, this yields a delta of the straddle of either −1 or 1.



cb) Vega:

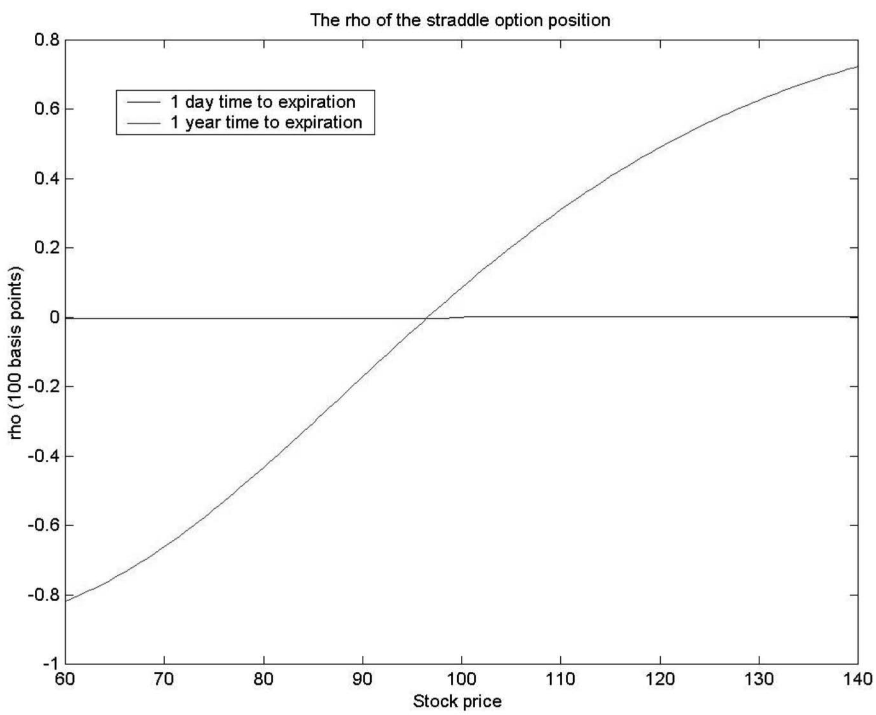
The one-day time to expiration vega graph shows only a small hump around the strike price of the option position. With only one day time to expiration left, we do not have enough time to participate in the opportunities the one percentage point increase in volatility offers to our bought straddle. However, with one year left, we see that the volatility increase has a huge effect on our straddle.

cc) Theta:



Remember, we bought a call and a put option on the same strike of $100. This figure is a nice demonstration that for bought at the money option positions, time decay is greatest for short time to maturity positions. Our long straddle will pay off if either the call or the put is in the money. If the current stock price is about 100 and we have only one day to expiration left, our option position will likely expire worthless. Therefore, there is a huge time decay. With longer time to maturities, chances of stock price movements away from a 100 are substantial. Therefore, the theta is much smoother and smaller.

cd) rho:

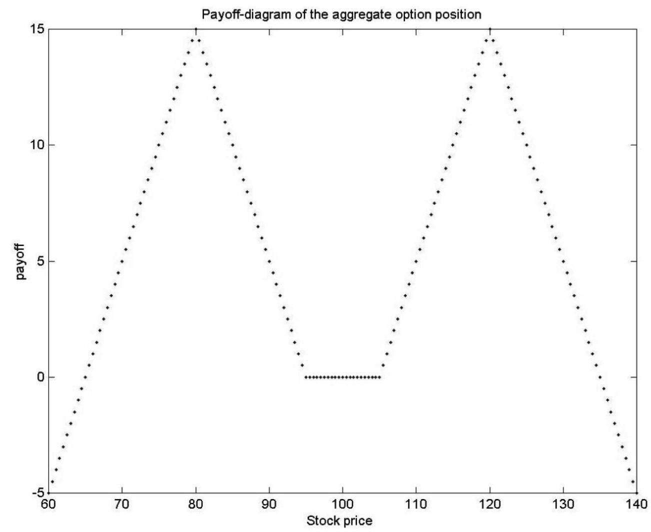


With one day to maturity left, a 100 basis point increase in the interest rate has no effect on the option position, because the time we could earn interest/lose interest on the strike is just too short.

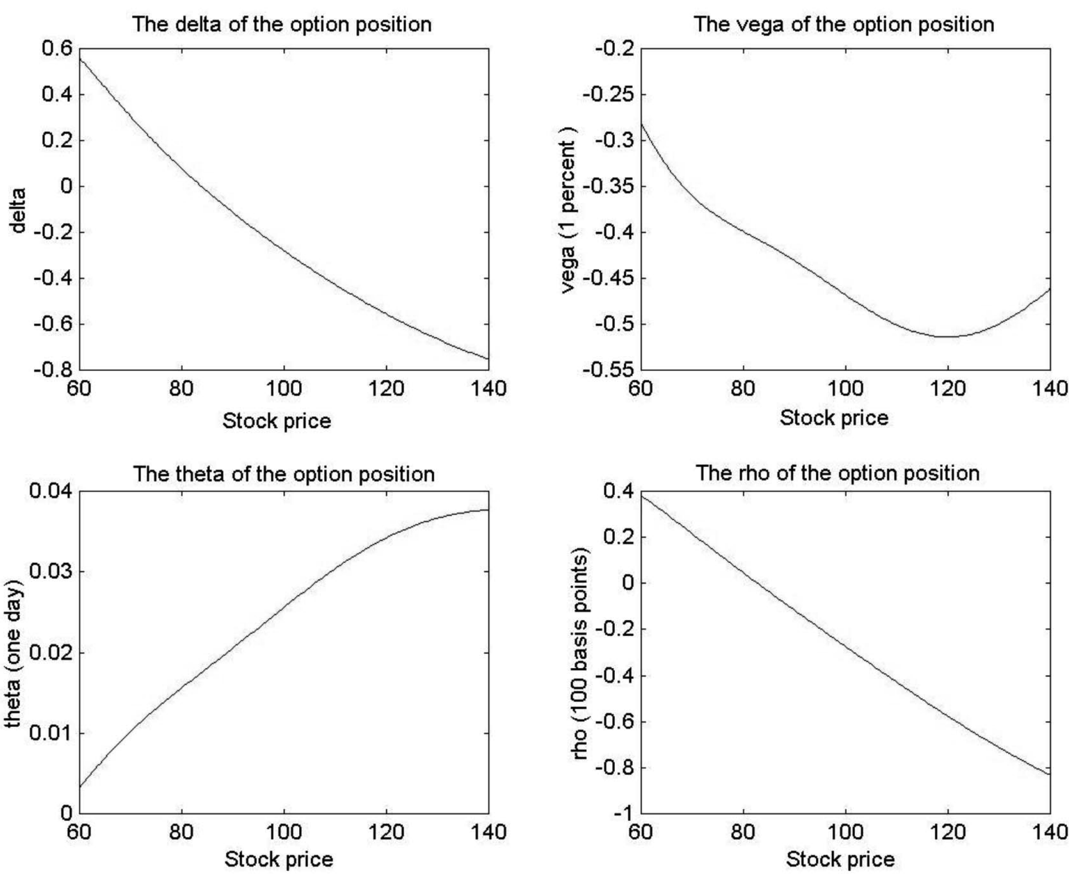
For the one year to maturity figure, we can see the following: If the stock price is higher than $100, it is the call option that is in the money, and the put expires worthless. Therefore, rho is positive (remember, rho for a call option is positive, because a call entails paying the fixed strike price to receive the stock and a higher interest rate reduces the present value of the strike). For stock prices smaller than $100, the put dominates and we know that the rho of a put is negative.

Question 12.17.

This is a figure of the payoff diagram of the option position:



From this position, the following greeks result:



Alternatively, just tabulate the greeks in \$5 stock price increases:

S	delta	vega	theta	rho
60	0.560	−0.281	0.003	0.381
65	0.430	−0.327	0.007	0.300
70	0.304	−0.360	0.010	0.215
75	0.186	−0.383	0.013	0.129
80	0.077	−0.399	0.016	0.045
85	−0.022	−0.414	0.018	−0.037
90	−0.115	−0.431	0.021	−0.117
95	−0.200	−0.449	0.023	−0.197
100	−0.281	−0.468	0.026	−0.275
105	−0.357	−0.486	0.028	−0.353
110	−0.428	−0.501	0.030	−0.429
115	−0.494	−0.511	0.032	−0.505
120	−0.556	−0.514	0.034	−0.577
125	−0.613	−0.510	0.036	−0.647
130	−0.665	−0.500	0.037	−0.713
135	−0.712	−0.483	0.037	−0.775
140	−0.754	−0.461	0.038	−0.832

Let's argue about the greeks from the standpoint of the options that are "active," i.e. that are in the money. Up to a stock price of 80, the two sold 80 put options and the bought 95 put options are active, with the two sold put options dominating. Therefore, the delta is initially positive. As we increase the stock price, the importance of the 80 puts decreases, and the 95 put (negative delta) and the 105 call (positive delta) become more important. As the stock price increases even further (say more than 95), the strong negative delta effect of the two sold 120 call options gradually takes over, dominating the positive delta effect of the active 105 call and ultimately pushing the delta down to -1.

As the rho for a put is negative and the rho for a call is positive, exhibiting the same decreasing respective increasing behavior in S as delta, the effects for rho work in the same way as for delta above.

This option position has the desirable feature of exhibiting a positive theta (i.e., as time to expiration gets closer, the option position value ceteris paribus increases. The sold options at the very low strike of 80 (puts) and the very high strike price of 120 (calls) are responsible for the positive theta. Theta is increasing in S because the time decay of the calls is higher (please refer to figures 12.6 and 12.8 in the text for an illustration).

For vega, the sold options dominate the aggregate vega position as in the theta case, making the vega negative. Remember that vega is highest if the option is at the money, so it makes sense that the vega becomes gradually more negative and has its minimum when the stock prices coincide with the strike price of the then dominating two sold call options.

Tables for 12.18. and 12.19.

Inputs			**Perpetual Options**	
Stock price	50		**Call**	**Put**
Exercise price	60	Option Price	26.35183	23.07471
Volatility	40.000%	Exercise at:	317.3092	22.6908
Risk-free interest rate	6.000%			
Dividend yield	3.000%			

Inputs			**Perpetual Options**	
Stock price	50		**Call**	**Put**
Exercise price	60	Option Price	22.75128	23.82482
Volatility	40.000%	Exercise at:	248.2475	21.75248
Risk-free interest rate	6.000%			
Dividend yield	4.000%			

Inputs			**Perpetual Options**	
Stock price	50		**Call**	**Put**
Exercise price	60	Option Price	27.10008	21.2744
Volatility	40.000%	Exercise at:	334.9193	25.08067
Risk-free interest rate	7.000%			
Dividend yield	3.000%			

Inputs			**Perpetual Options**	
Stock price	50		**Call**	**Put**
Exercise price	60	Option Price	29.83555	27.62938
Volatility	50.000%	Exercise at:	412.5475	17.45254
Risk-free interest rate	6.000%			
Dividend yield	3.000%			

Question 12.18.

a) The price of the perpetual call option is $26.35. It should be exercised when the stock price reaches the barrier of $317.31.

b) The price of the perpetual call option is now $22.75. It should be exercised when the stock price reaches the barrier of $248.25. The higher dividend yield makes it more costly to forego the dividends and wait for an increase in the stock price before exercising the option. Therefore, the option is worth less and it is optimal to exercise after a smaller increase in the underlying stock price.

c) The price of the perpetual call option is now \$27.10. It should be exercised when the stock price reaches the barrier of \$334.92. The higher interest rate increases the value of the call option and makes it attractive to wait a bit longer before you exercise the option, as you can continue to earn interest on the strike before you exercise. Therefore, the option is worth more and it is only optimal to exercise after a larger increase in the underlying stock price.

d) The price of the perpetual call option is \$29.84. It should be exercised when the stock price reaches the barrier of \$412.55. Options love volatility. The chances of an even larger increase in the stock price are high with a large standard deviation (and your risk is capped at the downside). Therefore, the option is worth more and you wait longer until you forego the future potential and exercise.

Question 12.19.

a) The price of the perpetual put option is \$23.07. It should be exercised when the stock price reaches the barrier of \$22.69.

b) The price of the perpetual put option is now \$23.82. It should be exercised when the stock price reaches the barrier of \$21.75. As the holder of the put option, you have the right to sell the underlying stock to somebody. Therefore, in your replication strategy, you are entitled to receive the dividends and you benefit from a higher dividend. The higher dividend yield makes it more desirable to wait for a larger decrease in the stock price before exercising the option. Therefore, the option is worth more and it is optimal to exercise after a larger decrease in the underlying stock price.

c) The price of the perpetual put option is now \$21.27. It should be exercised when the stock price reaches the barrier of \$25.08. The higher interest rate decreases the value of the put option, as you are entitled to earn interest on the strike price K once you exercised the option and obtained K from your counterparty. Therefore, waiting is more costly. The option is worth less and you exercise sooner.

d) The price of the perpetual put option is \$27.63. It should be exercised when the stock price reaches the barrier of \$17.45. Options love volatility. The chances of an even larger decrease in the stock price are high with a large standard deviation (and your risk of a stock price increase is capped). Therefore, the option is worth more and you wait longer until you forego the future potential and exercise.

Question 12.20.

a) $C(100, 90, 0.3, 0.08, 1, 0.05) = 17.6988$

b) $P(90, 100, 0.3, 0.05, 1, 0.08) = 17.6988$

c) The prices are equal. This is a result of the mathematical equivalence of the pricing formulas. To see this, we need some algebra. We start from equation (12.3) of the text, the formula for the European put option:

$$P(\bullet) = \overline{K} \times \exp(-\overline{r}T) \times N\left(-\frac{\ln\left(\frac{\overline{S}}{\overline{K}}\right) + \left(\overline{r} - \overline{\delta} - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \overline{S} \times \exp\left(-\overline{\delta}T\right)$$

$$\times N\left(-\frac{\ln\left(\frac{\overline{S}}{\overline{K}}\right) + \left(\overline{r} - \overline{\delta} + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Now we replace:

$$\overline{K} = S, \overline{r} = \delta, \overline{\delta} = r, \overline{S} = K$$

Then:

$$= S \times \exp(-\delta T) \times N\left(-\frac{\ln\left(\frac{K}{S}\right) + \left(\delta - r - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right) - K$$

$$\times \exp(-rT) \times N\left(-\frac{\ln\left(\frac{K}{S}\right) + \left(\delta - r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

Since $\ln\left(\frac{K}{S}\right) = -\ln\left(\frac{S}{K}\right)$

$$= S \times \exp(-\delta T) \times N\left(\frac{\ln\left(\frac{S}{K}\right) - \left(\delta - r - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right) - K \times \exp(-rT)$$

$$\times N\left(\frac{\ln\left(\frac{S}{K}\right) - \left(\delta - r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$= S \times \exp(-\delta T) \times N(d_1) - K \times \exp(-rT) \times N(d_2) = C(\bullet)$$

Question 12.21.

Inputs			**Perpetual Options**
Stock price	100		**Call**
Exercise price	90	Option Price	40.1589
Volatility	30.000%	Exercise at:	266.3405
Risk-free interest rate	8.000%		
Dividend yield	5.000%		

a) $C(100, 90, 0.3, 0.08, 1, 0.05) = \$40.16$

Exercise at \$ 266.34

Inputs			**Perpetual Options**
Stock price	90		**Put**
Exercise price	100	Option Price	40.1589
Volatility	30.000%	Exercise at:	33.79133
Risk-free interest rate	5.000%		
Dividend yield	8.000%		

b) $P(90, 100, 0.3, 0.05, 1, 0.08) = \$40.16$

We exercise at a price of \$33.79.

The prices are still identical. The ratio of the exercise barrier over the stock price for the call is equal to the inverse of the same ratio for the put option (2.66).