Chapter 14
Exotic Options: I

Question 14.1.

The geometric averages for stocks will always be lower.

Question 14.2.

The arithmetic average is 5 (three 5's, one 4, and one 6) and the geometric average is $(5 \times 4 \times 5 \times 6 \times 5)^{1/5} = 4.9593$. For the next sequence, the arithmetic average does not change (= 5); however the geometric average, $(3 \times 4 \times 5 \times 6 \times 7)^{1/5} = 4.7894$ is much lower. As the standard deviation increases (holding arithmetic means constant), the geometric return decreases. As an example, suppose we have two observations, $1 + \sigma$ and $1 - \sigma$. The arithmetic mean will be 1; however the geometric mean will be $\sqrt{(1 + \sigma)(1 - \sigma)} = \sqrt{1 - \sigma^2} < 1$.

Question 14.3.

Using the forward tree specification, $u = \exp(.08/2 + .3/\sqrt{2}) = 1.2868, d = \exp(.08/2 - .3/\sqrt{2}) = .84187$, and risk neutral probability $p = (e^{.08/2} - d)/(u - d) = .44716$. The two possible prices in 6 months are 128.68 and 84.19; the three possible 1 year prices are 165.58, 108.33, and 70.87.

a) Using the 6m and 12m prices, the possible arithmetic averages (in 1 year) are 147.13, 118.50, 96.26, and 77.53. The four possible geometric averages are 145.97, 118.07, 95.50, and 77.24.

b) Since we are averaging the 6m and 12m prices, the average tree will be identical for the current node and the two 6m nodes; however, the 12m node will have the four nodes given in the previous answer.

c) With $K = 100$ the up-up value is 47.1266 and the up-down value is 18.5026. There is no value in the bottom half of the tree. This gives an up value of $e^{-.04}\,(p47.1266 + (1 - p)\,18.5026) = 30.075$ and an initial value of $e^{-.04}p30.075 = 12.921$.

d) Similarly, up value is $e^{-.04}\,(p45.9652 + (1 - p)\,18.0651) = 29.344$ and an initial value of $e^{-.04}p29.344 = 12.607$.

Question 14.4.

Using the forward tree specification, $u = \exp(.08/2 + .3/\sqrt{2}) = 1.2868, d = \exp(.08/2 - .3/\sqrt{2}) = .84187$, and risk neutral probability $p = (e^{.08/2} - d)/(u - d) = .44716$. The two possible prices in 6 months are 128.68 and 84.19; the three possible 1 year prices are 165.58, 108.33, and 70.87.

Using the 6m and 12m prices, the possible arithmetic averages are (in 1 year) are 147.13, 118.50, 96.26, and 77.53. The four possible geometric averages are 145.97, 118.07, 95.50, and 77.24. These are in the order: u-u, u-d, d-u, and d-d.

a) The four intrinsic values will be $165.58 - 147.13 = 18.45$ (u-u), 0 (u-d), $108.33 - 96.26 = 12.07$ (d-u), and zero (d-d). This will give an up value of $e^{-.04}p18.45 = 7.93$, a down value of $e^{-.04}p12.07 = 5.19$, and an initial value of $e^{-.04}(p7.93 + (1-p)5.19) = 6.1602$.

b) The four intrinsic values will be $165.58 - 145.97 = 19.61$ (u-u), zero (u-d), $108.33 - 95.50 = 12.83$ (d-u), and zero (d-d). This will give an up value of $e^{-.04}p19.61 = 8.43$ and a down value of $e^{-.04}p12.83 = 5.51$ with an initial value of $e^{-.04}(p8.43 + (1-p)5.51) = 6.55$.

Question 14.5.

See Table One.

Table One (Problem 14.5)

		Price Average		Payoff	
	RN Prob	Arith	Geo	Arith	Geo
uuu	0.0954	151.1369	149.1442	51.1369	49.1442
uud	0.1134	133.3612	132.8796	33.3612	32.8796
udu	0.1134	118.8058	118.3887	18.8058	18.3887
udd	0.1348	106.2345	105.4781	6.2345	5.4781
duu	0.1134	106.8874	105.4781	6.8874	5.4781
dud	0.1348	94.3160	93.9754	0	0
ddu	0.1348	84.0221	83.7272	0	0
ddd	0.1602	75.1314	74.5965	0	0
		Expected RN Payoff		12.4116	11.8580
		Discounted		**11.4574**	**10.9463**

Question 14.6.

a) A standard call is worth 4.1293.

b) A knock in call will also be worth 4.1293 (you can verify this with the software). In order for the standard call to ever be in the money, it must pass through the barrier. They therefore give identical payoffs.

c) Similar reasoning, implies the knock-out will be worthless since in order for $S_T > 45$, the barrier must have been hit making knocking out the option.

Question 14.7.

See Table Two for the prices and ratio. The longer the time to expiration, the greater the dispersion of S_T. For the standard call option, this unambiguously increases the value (by standard convexity arguments). For the knock-out, there is a trade-off. The higher the dispersion, the greater chance for large payoffs; however, there will also be a higher chance for the barrier to be hit.

Table Two (Problem 14.7)

T	Standard	Knock-Out	Ratio
0.25	0.9744	0.7323	1.3306
0.5	2.1304	1.2482	1.7067
1	4.1293	1.8217	2.2667
2	7.4398	2.4505	3.0360
3	10.2365	2.8529	3.5881
4	12.6969	3.1559	4.0232
5	14.9010	3.4003	4.3823
100	39.9861	5.3112	7.5286

Question 14.8.

See Table Three for the prices and ratio. The longer the time to expiration, the greater the dispersion of S_T. For the standard put option, this increases the value unless the option expiration starts to become large (we lose time value of receiving the strike price). For the knock-out, there is an extra negative effect a higher expiration date has. With higher dispersion of S_T, the greater chance for larger S_T, the greater the chance of being knocked out; however, there will also be a higher chance for the barrier to be hit.

Table Three (Problem 14.8)

T	Standard	Knock-Out	Ratio
0.25	5.0833	3.8661	1.3148
0.5	5.3659	3.4062	1.5753
1	5.6696	2.8626	1.9806
2	5.7862	2.2233	2.6025
3	5.6347	1.8109	3.1115
4	5.3736	1.5094	3.5601
5	5.0654	1.2761	3.9695
100	0.0012	0.0001	24.4951

Question 14.9.

See Table Four on the next page for the values. This highlights the trade-off increasing the time to maturity has on the knock out call option. When time to maturity increases, the standard call has the interest on the strike as well as the higher dispersion of S_T making it more valuable. For the knock out call, the likelihood of getting knocked out can offset this effect.

Table Four (Problem 14.9)

Months	Black-Scholes	U & O Call	Ratio
1	0.1727	0.1727	1.0003
2	0.5641	0.5479	1.0296
3	0.9744	0.8546	1.1401
4	1.3741	1.0384	1.3233
5	1.7593	1.1243	1.5649
6	2.1304	1.1468	1.8577
7	2.4886	1.1316	2.1991
8	2.8353	1.0954	2.5885
9	3.1718	1.0482	3.0260
10	3.4991	0.9962	3.5124
11	3.8180	0.9430	4.0488
12	4.1293	0.8906	4.6365

Question 14.10.

When $K = 0.9$, the only scenarios where the up and out puts have a different payoff than the standard put is where the exchange rate rises to the barrier of 1 (or 1.05) before six months (i.e. $x_t > 1$ for $t < T$) and then end below .9 (i.e. $x_T < .9$). In this case, the up and out puts will pay nothing (they will have gotten knocked out) and the standard put will pay the intrinsic value $.9 - x_T$. Given the volatility assumption, these scenarios are virtually impossible and, for the small chance that they happen, the payoff in for the standard put would be small.

When $K = 1$ the scenarios mentioned above are much more likely as x only has to rise above 1 (or 1.05) and then finish below 1. With higher time to expirations, the probabilities of such scenarios will become non negligible and we should expect the up and out to have lower values than the standards (when $K = .9$).

Question 14.11.

a) 9.61

b) In one year, the option will be worth more than \$2 if $S_1 > 31.723$.

c) 7.95

d) If we buy the compound call in part b) and sell the compound option in this question for x we will be receiving the standard call in one year for \$2 regardless of S_1. Hence, our total cost is $7.95 - x + 2e^{-.08} = 9.61$, which implies $x = .18623$. Without rounding errors it would be .18453.

Question 14.12.

a) 3.6956

b) In one year, the put option will be worth more than \$2 if $S_1 < 44.35$.

c) 2.2978

d) If we buy the standard put from part a) as well as this compound option for x we will keep the standard put if $S_1 < 44.35$ and sell it for \$2 otherwise. This identical to putting $2e^{-.08}$ in the risk free bond and buying the compound option in part c). The total costs must be identical implying $3.6956 + x = 2.2978 + 2e^{-.08}$, implying $x = .448$.

Question 14.13.

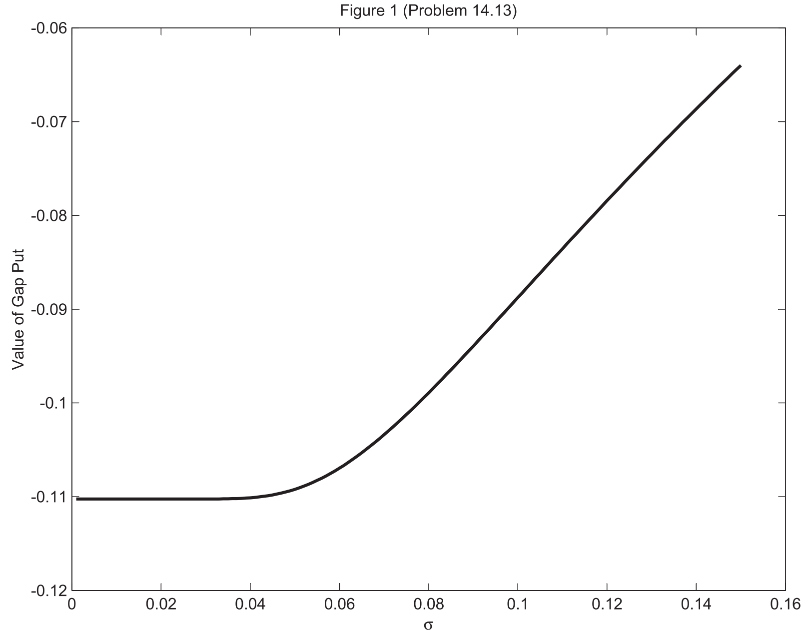
a) $P(S, K_1, K_2, \sigma, r, T, \delta) = K_1 e^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$ where d_i are the same as equation (14.15). For foreign currency, $\delta = r_E$, $S = x$, and $r = r_\$$.

b) A gap put will pay $.8 - x$ when $x < 1$. With zero volatility, $x = x_0 = .9$ and this will mean we will be selling the foreign currency for .8 dollars. This is equivalent to a forward contract with delivery price $K_1 = .8$,

$$f = -.9e^{-.03/2} + .8e^{-.06/2} = -.11024. \tag{1}$$

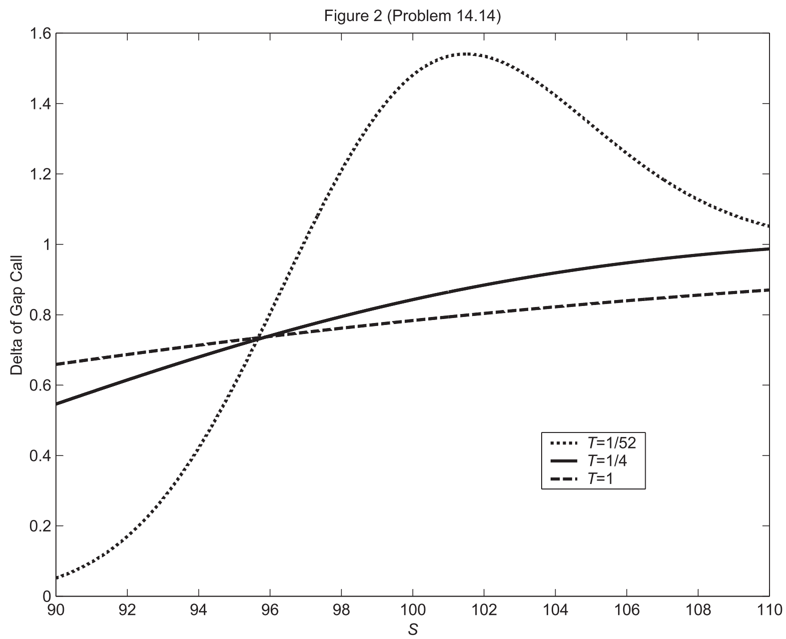
If volatility increases we will have the potential for upside if x can fall; this will be offset only for x rising up to $K_2 = 1$. If $x_T > 1$, we get the discontinuous jump from losing $-.2$ to having zero liability. This asymmetry should have the gap put becoming more valuable as volatility increases. Figure 1 confirms this.

Figure 1 (Problem 14.13)



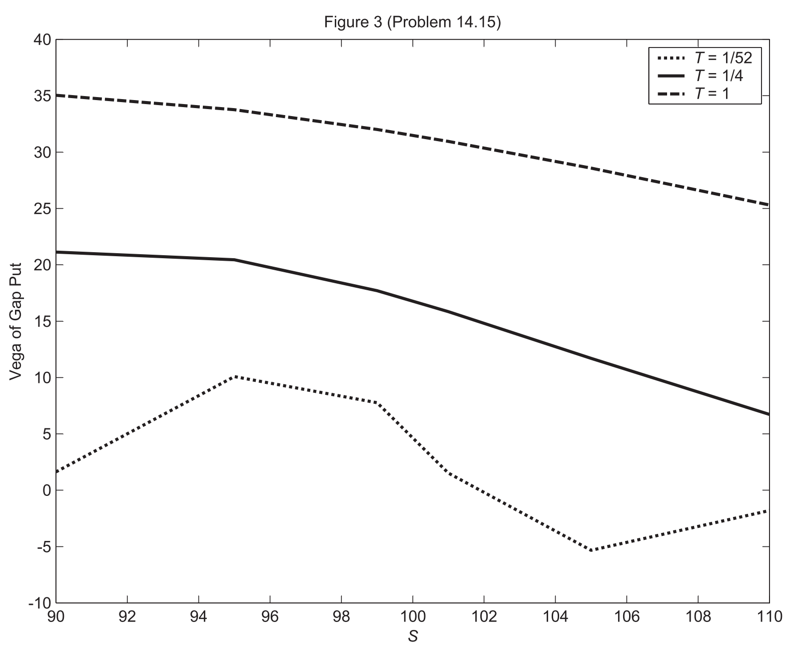
Question 14.14.

Using $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. See Figure Two on the next page. When we are close to maturity (e.g. $T = 1/52$) we see large variations in delta. The discontinuity at K_2 can require deltas greater than one. The value of the option can go from close to zero to close to $10 with little movement in the price (if S_T is close to K_1). If $T \simeq 0$, delta will be close zero for $S < 100$, enormous for $S = 100$, and close to one if $S > 100$. This problem does not occur as T becomes larger.



Question 14.15.

Using $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. See Figure Three. For three month and one year gap put



options, the option value increases with volatility (the value function is convex in S). When $T = 1/52$, if $S > 100$ the option loses value with higher volatility due to the increased likelihood of a negative payoff (the value function is concave in S).

Question 14.16.

Under Black Scholes the standard 40-strike call on S will be

$$BSCall\,(40, 40, .3, .08, 1, 0)\,. \tag{2}$$

For the exchange option on S using $2/3$ of a share of Q as the strike, we use a strike of $(2/3)\,60 = 40$, a volatility of $\sqrt{.3^2 + .5^2 - 2\,(.5)\,(.3)\,(.5)} = .43589$, and an "interest rate" of .04:

$$BSCall\,(40, 40, .43589, .04, T, 0)\,. \tag{3}$$

For, all but very long time to maturities, the higher volatility will offset the lower "interest" and the exchange option will be worth more. With $T = 1$, we have the standard option is worth 6.28 and the exchange option is worth 7.58.

Question 14.17.

We use 1 year options.

a) The price of falls from 2 to 1.22 as we increase the dividend yield of S from 0 to .1.

b) The price rises from 2 to 2.86 as we increase the dividend yield of Q from 0 to .1.

c) The price falls from 5.79 to 2 as we increase the correlation from $-.5$ to .5.

d) Standard arguments for δ. As δ_Q increases, the yield on the strike asset makes delaying our purchase of S more valuable (the same as why a higher r makes standard call options more valuable). As ρ increases the volatility of the *difference* goes down (in this case from 70% to 43%).

Question 14.18.

a) $Var\,[\ln{(S/Q)}] = .3^2 + .3^2 - 2\,(.3)\,(.3) = 0$ and the option is worthless (it will never be in the money as $S_T = Q_T$).

b) $Var\,[\ln{(S/Q)}] = .3^2 + .4^2 - 2\,(.3)\,(.4) = .01$ hence we use a 10% volatility in Black-Scholes. With $T = 1$ we have the exchange option equal to \$1.60.

c) If $\ln(S)$ and $\ln(Q)$ are jointly normal with $\rho = 1$ then they are linearly related. Hence

$$\ln(Q) = \ln(40)\left(1 - \frac{\sigma_Q}{\sigma_S}\right) + \frac{\sigma_Q}{\sigma_S}\ln(S). \tag{4}$$

In question a), $\ln(Q) = \ln(S) \Longrightarrow Q = S$. For question b),

$$\ln(Q) = -\frac{\ln(40)}{3} + \frac{4}{3}\ln(S) \Longrightarrow Q = .2924S^{4/3}. \tag{5}$$

If S rises (say $S_T = 50$) then Q will be greater than S (say $Q_T = 53.861$); the option will be in the money if S falls for Q will fall by a greater amount making the exchange option have value.

Question 14.19.

XYZ will have a "natural hedge" when x (\$ price of Euro) and S (\$ price of oil) move in together. For example, if x rises (the Euro appreciates implies "good news" for XYZ) and S rises ("bad news" for XYZ) the two risks offset. Similarly if x falls and S falls. When the two move opposite, the company is either "win-win" ($x \uparrow$ and $S \downarrow$) or "lose-lose" ($S \uparrow$ and $x \downarrow$). An exchange option paying $S - x$ is therefore natural for XYZ. They will give up upside to hedge against downside. This is likely to be cheaper than treating the two risks separately due to $\rho > 0$ implying the exchange option will have a lower (implied) volatility.

Question 14.20.

a) Since the options will be expiring at t_1, we have the payoff of a put if $S_T < K$ and the payoff of a call if $S_T > K$. This is equivalent to a K strike straddle.

b) Using put-call parity at t_1, the value of the as-you-like-it option at t_1 will be:

$$\max\left(C(S_1, K, T - t_1), C(S_1, K, T - t_1) + Ke^{-r(T-t_1)} - Se^{-\delta(T-t_1)}\right) \tag{6}$$

$$= C(S_1, K, T - t_1) + \max\left(0, Ke^{-r(T-t_1)} - Se^{-\delta(T-t_1)}\right) \tag{7}$$

$$= C(S_1, K, T - t_1) + e^{-\delta(T-t_1)}\max\left(0, Ke^{(\delta-r)(T-t_1)} - S\right). \tag{8}$$

The first term is the value of a call with strike K and maturity T; the second term is the payoff from holding $e^{-\delta(T-t_1)}$ put options that expire at t_1 with strike $Ke^{(\delta-r)(T-t_1)}$.

Question 14.21.

a) In 6 months, a 3 month at-the-money call option will be worth 6.9618 if $S = 100$, 3.4809 if $S = 50$, and 13.9237 if $S = 200$. Note it is always 6.9618% of the stock price.

b) In six months ($t_1 = 1/2$), we will need $.069618S_T$; this can be done by buying .069618 shares of stock (since there are no dividends).

c) We should pay \$6.9618 for the forward start (the cost of the shares); this is the same as the current value of a 3m at-the-money option.

d) Using similar arguments, a 3m 105% strike is always worth 4.7166% of the stock price. We should then pay \$4.7166 for a forward start 105%-strike option.

Question 14.22.

a) \$6.0831

b) The current price of a 1m 95-strike put is 1.2652. In fact, a 1m put with a strike equal to 95% of the stock price will always be equal to 1.2652% of the stock price. Therefore, the present value of twelve of these 1m 95% strike puts is $12\,(1.2652) = 15.182$.

c) Technically, and perhaps non-intuitively, the rolling insurance strategy costs more because it is more expensive to replicate. Note that one strategy doesn't dominate another. If the price never falls less than 5% in month, all of the 12 one month options will be worthless; yet the price in 1 year could have fallen by more than 5%. Interest aside, the rollover options will give the holder $\sum_{i=0}^{11} \max\,(S_{i+1} - .95S_i, 0)$; whereas, the simple insurance gives the holder $\max(S_{12} - .95S_0, 0)$. The rollover strategy has the advantage of being able to provide payoffs (insurance) for each month regardless of the past. If the stock price rises in one month to (say) \$120, the simple insurance option will be less effective whereas the rollover will provide a new insurance option with a strike of $.95\,(120) = 114$.