Chapter 18
The Lognormal Distribution

Question 18.1.

The five standard normals are $\frac{-7+8}{\sqrt{15}} = .2582$, $\frac{-11+8}{\sqrt{15}} = -.7746$, $\frac{-3+8}{\sqrt{15}} = 1.291$, $\frac{2+8}{\sqrt{15}} = 2.582$, and $\frac{-15+8}{\sqrt{15}} = -1.8074$.

Question 18.2.

If z is standard normal, $\mu + \sigma \times z$ is $\mathcal{N}\left(\mu, \sigma^2\right)$ hence our five standard normals can be use to create the desired properties: $.8 + 5\,(-1.7) = -7.7$, $.8 + 5\,(.55) = 3.55$, $.8 + 5\,(-.3) = -0.7$, $.8 + 5\,(-.02) = .7$, and $.8 + 5\,(.85) = 5.05$.

Question 18.3.

$x_1 + x_2$ is normally distributed with mean -1 and variance $5 + 2 + 2(1.3) = 9.6$. $x_1 - x_2$ is normally distributed with mean 3 and variance $5 + 2 - 2(1.3) = 4.4$.

Question 18.4.

Sums and differences of two random variables are normally distributed hence $x_1 + x_2$ is normally distributed with mean $\mu_1 + \mu_2 = 10$ and variance

$$\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2 = 0.5 + 14 + \left[2 \times (-0.3) \times \sqrt{0.5 \times 14}\right] = 10.3$$

The difference is normally distributed with mean $\mu_1 - \mu_2 = -6$ and (higher) variance

$$\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = 0.5 + 14 - \left[2 \times (-0.3) \times \sqrt{0.5 \times 14}\right] = 18.7.$$

Question 18.5.

$x_1 + x_2 + x_3\tilde{}\mathcal{N}\,(5.5, 47.8)$, $x_1 + 3x_2 + x_3\tilde{}\mathcal{N}\,(9.5, 123.4)$, and $x_1 + x_2 + .5x_3\tilde{}\mathcal{N}\,(4.25, 30.65)$.

Question 18.6.

If $x\tilde{}\mathcal{N}\left(\mu, \sigma^2\right)$ then $E\left(e^x\right) = ex^{\mu+.5\sigma^2}$; using the given numbers, $E\left(e^x\right) = e^{2+2.5} = 90.017$. There is a 50% probability x is below its mean of 2 hence the median of e^x is $e^2 = 7.3891$.

Question 18.7.

Denote the stock price of month i by S_i and let the continuous return of month i be denoted as $R_i = \ln(S_{i+1}/S_i)$.

a) The average of the monthly continuous returns is zero for both stock A and B; the annual return is also zero. Note the average simple return will be much higher for stock B although both stocks have had the same 4 month holding period return.

b) The standard deviations, since $\overline{R} = 0$, is $\frac{1}{4}\sum_i R_i^2$. For A, this monthly standard deviation is 0.167% which is a yearly standard deviation of $\sqrt{12}\,(.00167) = .578\%$. For B, the monthly standard deviation is 8.014% which is a yearly standard deviation of 27.76%. For this example we did not adjust for degrees of freedom (i.e. divide the sum by 3 instead of 4).

c) No matter what time unit we use, the (annualized) mean continuous return will be $\ln(S_T/S_0)/T$. This is due to the intermediate observations cancelling. For example, suppose we have 2 years of data observed at the end of the month (for 24 months of data). The mean monthly continuous return is

$$\frac{1}{24}\sum_{i=1}^{24}\ln(S_{i+1}/S_i) = \frac{1}{24}\sum_{i=1}^{24}\ln(S_i) - \ln(S_{i-1}) = \frac{1}{24}\left[\ln(S_{24}) - \ln(S_0)\right].$$

The intermediate end of month prices are irrelevant. This average, when annualized (i.e. multiply the above by 12), is the same average we would get if we would use the average of two yearly continuous returns. For standard deviation, there will be squared deviations which lead to cross terms. The sampling interval will, therefore, give different estimates of volatility.

Question 18.8.

Since $S_t = S_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z\right)$ where z is standard normal,

$$\begin{aligned}
P(S_t > 105) &= P\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z > \ln\left(\frac{105}{100}\right)\right) \\
&= P\left(z > \frac{\ln\left(\frac{105}{100}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \\
&= P\left(-z < \frac{-\ln\left(\frac{105}{100}\right) + \left(\alpha - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) = N(d_2)
\end{aligned}$$

where $d_2 = \left[\ln(100/105) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right] / \left(\sigma\sqrt{t}\right)$. Using the given parameters, $d_2 = .045967$and $N(d_2) = .4817$. For this parameter specification, the probability $S_t > 105$ increases with t and decreases with σ. Analytically, since $N'(d_2) > 0$, the derivative will have the same sign as $\partial d_2/\partial t$ and $\partial d_2/\partial\sigma$. Specifically,

$$\frac{\partial P(S_t > K)}{\partial t} = N'(d_2)\frac{\left(\alpha - \sigma^2/2\right) - \ln(S_0/K)/t}{2\sigma\sqrt{t}} > 0$$

since $\alpha - \sigma^2/2 = .035 > 0$ and $\ln(S_0/K) < 0$. As an example, if t is 5 years, there is a 57.46% chance of being greater than 105. For volatility, let $t = 1$. Then

$$\frac{\partial P(S_t > K)}{\partial\sigma} = -N'(d_2)\left(\frac{\alpha - \ln(S_0/K)}{2\sigma^2} + \frac{1}{4}\right) < 0.$$

Question 18.9.

We have $d_1 = .2540$, and using equation (18.30),

$$E(S_1|S_1 > 105) = 100e^{.08}\frac{N(d_1)}{N(d_2)} = 100e^{.08}\frac{.6003}{.4817} = 135.$$

Table One shows how changes in t and σ affect this conditional expectation. We see it increases with time and volatility.

Table One (Problem 18.9)

t	$E(S_t\|S_t>105)$	σ	$E(S_1\|S_1>105)$	r	$E(S_1\|S_1>105)$
0.25	117.19	0.10	115.15	0.01	131.63
0.50	123.98	0.20	124.65	0.02	132.08
0.75	129.74	0.30	135.00	0.03	132.54
1.00	135.00	0.40	146.25	0.04	133.01
1.25	139.97	0.50	158.48	0.05	133.49
1.50	144.76	0.60	171.78	0.06	133.98
1.75	149.42	0.70	186.26	0.07	134.49
2.00	154.01	0.80	202.04	0.08	135.00
2.25	158.54	0.90	219.26	0.09	135.53
2.50	163.05	1.00	238.06	0.1	136.07
2.75	167.54	1.10	258.61	0.11	136.63
3.00	172.04	1.20	281.11	0.12	137.19
3.25	176.54	1.30	305.77	0.13	137.78
3.50	181.06	1.40	332.82	0.14	138.37
3.75	185.61	1.50	362.55	0.15	138.98
4.00	190.19	1.60	395.25	0.16	139.60
4.25	194.81	1.70	431.27	0.17	140.24
4.50	199.48	1.80	471.02	0.18	140.90
4.75	204.19	1.90	514.93	0.19	141.57
5.00	208.95	2.00	563.51	0.2	142.25

Question 18.10.

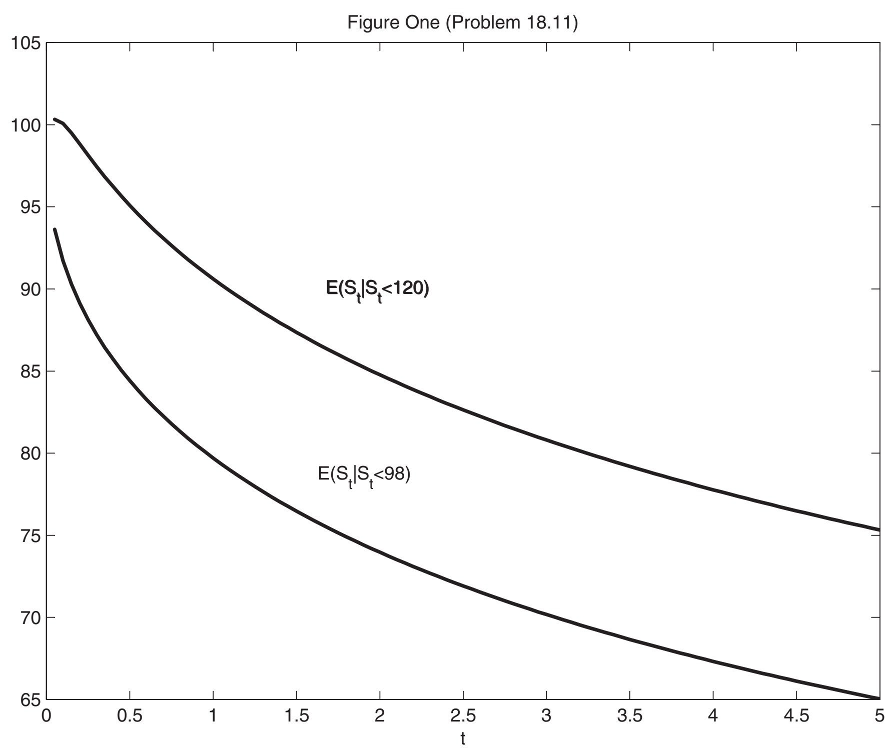
We have

$$P\left(S_t < 98\right) = P\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}z < \ln\left(\frac{98}{100}\right)\right)$$
$$= P\left(z < \frac{\ln\left(\frac{98}{100}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) = N\left(-d_2\right)$$

with $-d_2 = \left(\ln\left(98/100\right) - .035\right)/.3 = -.18401$. Hence $P\left(S_t < 98\right) = N\left(-.18401\right) = 42.70\%$.

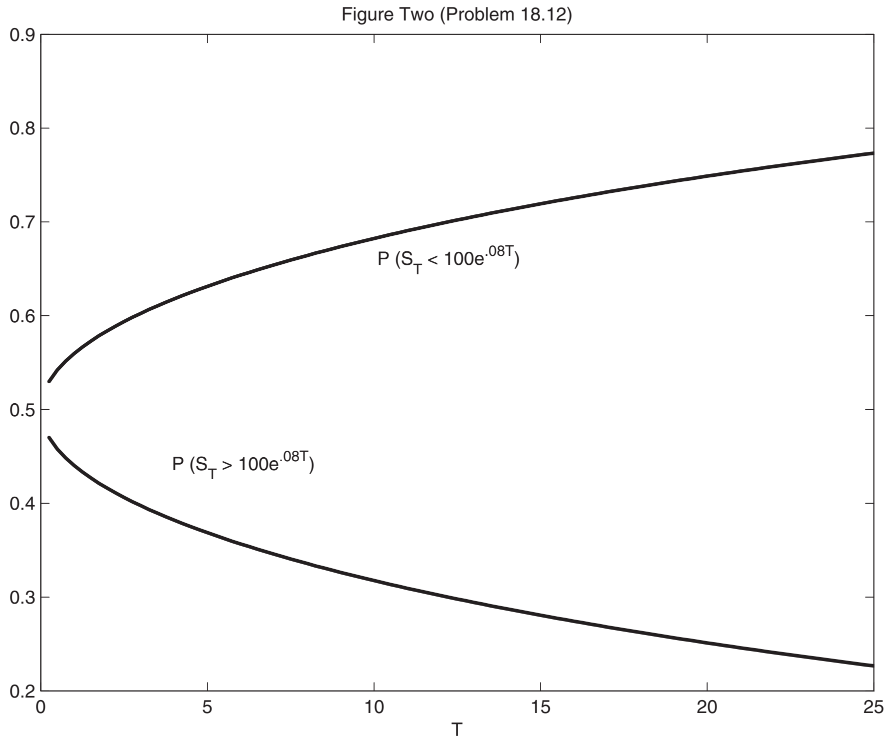
Question 18.11.

Using equation 18.28, $E(S_1|S_1 < 98) = 100e^{.08}N(-.484)/N(-.184) = 79.71$. Similarly, $E(S_1|S_1 < 120) = 100e^{.08}N(.1911)/N(.4911) = 90.62$. See Figure One for the negative change in both expectations as we increase t. When $\sigma = .1$, $E(S_1|S_1 < 98) = 92.99$ and $E(S_1|S_1 < 120) = 105.35$. Both significantly increase.



Question 18.12.

See Figure Two on the next page. Option prices depend on the conditional (risk neutral) expectation, not the probability the option is in the money. As T increases, the likelihood that $S_T > K_T$ may be lower; however, the payoff depends on the conditional expectation (since the option does not pay a constant amount). The increased dispersion offsets the lower probability (for the call option).



Question 18.13.

An example is with $K = 80$. When t is less than approximately 6 years, $P(S_t < 80)$ is increasing in t; however, if t is larger, the probability decreases. For example, if $t = 1/12$ we have $P\left(S_{1/12} < 80\right) = 0.45\%$. If we increase the time by one month (i.e. $t = 2/12$) $P\left(S_{1/6} < 80\right) = 3.08\%$. When $t = 10$, $P(S_{10} < 80) = 27.29\%$; with $t = 10 + 1/12$, $P\left(S_{121/12} < 80\right) = 27.27\%$. The effect on the conditional expectation is unambiguous (it decreases).

Question 18.14.

The mean should be varying year by year; whereas, the standard deviation should be more stable.

Question 18.15.

Although both data should appear non (log) normal, the weekly data should be closer to normality.