Chapter 19
Monte Carlo Valuation

Question 19.1.

The histogram should resemble the uniform density, the mean should be close to 0.5, and the standard deviation should be close to $1/\sqrt{12} = 0.2887$.

Question 19.2.

The histogram should be similar to a standard normal density ("bell" shaped). Since a uniform distribution has a mean of 0.5 and a variance of 1/12, the mean of $\sum_{i=1}^{12} u_i - 6$ is zero and the variance (& standard deviation) will be one since

$$var\left(\sum_{i=1}^{12} u_i\right) = 12var\left(u_i\right) = 1.$$

Question 19.3.

The mean of e^{x_1} should be close to $e^{1/2} = 1.6487$ and the mean of e^{x_2} should be close to $e^{.7+1.5} = 9.025$.

Question 19.4.

The standard deviation of the estimate will be $s_n/\sqrt{n}$ where s_n is the sample standard deviation of the n simulations. Since s_n is close to 2.9, $n = 84000$ should give a standard error close to 0.01.

Question 19.5.

$1/S_1 = \exp\left(-.08 + .3^2/2 + .3z\right)/40$ generates the simulations. The mean of which should be close to $e^{-.035+.3^2/2}/40 = .02525$. This should also be the forward price.

Question 19.6.

The simulations should be generated by $S_1 = 100\exp\left(.06 - .4^2/2 + .4z\right)$ where z is standard normal. The claim prices should be $e^{-.06}\overline{S^\alpha}$ where α is the relevant power and the $\overline{S^\alpha}$ is the average from the simulations. These values should be close to

$$100^\alpha \exp\left((\alpha - 1)\left(.06 + \frac{\alpha}{2}.4^2\right)\right).$$

Using this, the three values should be close to 12461, 9.51, and .000135 respectively.

Question 19.7.

The five values should be close to 10366.56, 1.004, 96.95, 10^{-4}, and 1, 261, 120 respectively.

Question 19.8.

By log normality

$$P\left(S_t < 95\right) = P\left(100 \exp\left(\left(.1 - .2^2/2\right)t + .2\sqrt{t}z\right) < 95\right)$$
$$P\left(z < \frac{\ln\left(95/100\right) - \left(.1 - .2^2/2\right)t}{.2\sqrt{t}}\right)$$

with $t = 1/365$ this is $N(-4.9207) = 4 \times 10^{-7}$. This magnitude negative return should, on average, occur once every 2.5 million days. With $t = 1/252$ (i.e. one trading day) this becomes $N(-4.0965) = 2.097 \times 10^{-5}$; making such a drop is similarly unlikely.

Question 19.9.

See Figure One for a typical histogram of a mean zero return and Figure Two for a normal plot with different high volatility probabilities.

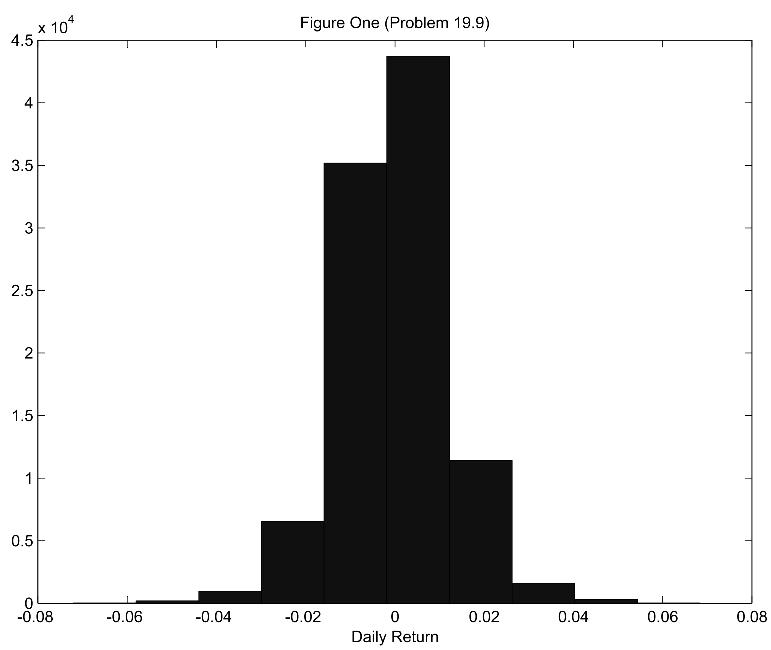
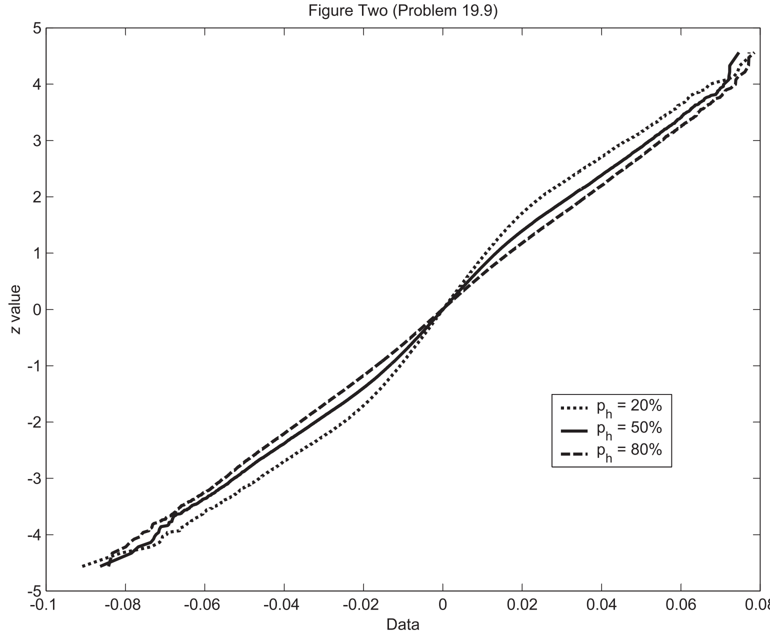


Figure Two (Problem 19.9)



Question 19.10.

The simulations should be done by generating 1000 standard normals ε_1 and another 1000 (independent) standard normals ε_2. Then let $S_1 = 40\exp\left(\frac{.08-.3/2}{12} + \frac{.3}{\sqrt{12}}\varepsilon_1\right)$ and let $S_2 = 100\exp\left(\frac{.08-.5/2}{12} + \frac{.5}{\sqrt{12}}z\right)$ where $z = .45\varepsilon_1 + \sqrt{1-.45^2}\varepsilon_2$. The means, standard deviations, and correlation from Monte Carlo should approximate their theoretical counterparts.

Question 19.11.

The estimate should be within a few cents of the true value. The standard deviation of the estimate should be .015 with a 95% confidence interval of $[\widehat{p} - .015, \widehat{p} + .015]$. Using naive control variate with a call option should add considerable noise to the estimate. For the adjusted control variate, β should be close to .0161. The variance of this estimate will be lowered from approximately 22% to 14%.

Question 19.12.

See Table One for a typical simulation. With weekly data, the lognormality (as opposed to normal)

TABLE ONE (Problem 19.12)

	Returns		Stock Price	
	Week	Year	Week	Year
Mean	0.0030	0.1551	100.3862	122.1316
SD	0.0415	0.2995	4.1708	37.4255
Skewness	0.0013	0.0013	0.1267	0.9511
Kurtosis	3.0133	3.0133	3.0390	4.6230

of the stock price isn't strong (i.e. there is little skewness or kurtosis). This implies the simple return is not that different from the continuous (normal) return. However, when we look at yearly distributions, there is now significant kurtosis and skewness.

Question 19.13.

See Table Two for a typical simulation (note these are gross simple returns). We should expect significant skewness and kurtosis due to many returns of 0% (i.e. out of the money) being offset by a few very large returns.

TABLE TWO (Problem 19.13)

	Call	Put
Mean	2.1910	0.6700
SD	2.6096	1.3406
Skewness	1.6207	2.2302
Kurtosis	6.3951	7.5730

Question 19.14.

Using simple gross returns (i.e. $Payoff/Cost$), the mean should be around 1.62, the standard deviation around 1.49, skewness around 1.79, and kurtosis around 7.59.

Question 19.15.

We need to calculate the discounted continuation value. The payoff of the option at $t = 3$ is equal to $\max(K - S(3), 0)$. The PV of the continuation value at $t = 2$ is Payoff ($t = 3$) $*$ $\exp(-0.06)$.

S0	1
K	1.1
R	0.06

Path	t=0	t=1	Payoff t=1	t=2	Payoff t=2	PV(continuation)	t=3	Payoff t=3
1	1	1.09	0.01	1.08	0.02	0	1.34	0
2	1	1.16	0	1.26	0		1.54	0
3	1	1.22	0	1.07	0.03	0.065923517	1.03	0.07
4	1	0.93	0.17	0.97	0.13	0.169517616	0.92	0.18
5	1	1.11	0	1.56	0		1.52	0
6	1	0.76	0.34	0.77	0.33	0.188352907	0.9	0.2
7	1	0.92	0.18	0.84	0.26	0.084758808	1.01	0.09
8	1	0.88	0.22	1.22	0		1.34	0

It is important to only include the paths at $t = 2$ in the regression where the put option is in the money:

Regression

Path	t=2	S^2	Payoff t=2	PV(continuation)
1	1.08	1.1664	0.02	0
3	1.07	1.1449	0.03	0.065924
4	0.97	0.9409	0.13	0.169518
6	0.77	0.5929	0.33	0.188353
7	0.84	0.7056	0.26	0.084759

S^2	S	Intercept
−1.81	**2.98**	**−1.07**
3.623	6.750	3.094
0.54564	0.073867	#N/A

For period $t = 1$, we have to do very similar calculations. We calculate the continuation value and the present value in $t = 1$ of that continuation value, estimate a regression, and then compare the fitted continuation value with the value of immediate exercise. We have:

Path	t=0	t=1	Payoff t=1	PV(continuation value)	Payoff t=2	Conditional Payoff t=3
1	1	1.09	0.01	0	0	0
2	1	1.16	0	0	0	0
3	1	1.22	0	0.062084431	0	0.07
4	1	0.93	0.17	0.122429389	0.13	0
5	1	1.11	0	0	0	0
6	1	0.76	0.34	0.310782296	0.33	0
7	1	0.92	0.18	0.244858779	0.26	0
8	1	0.88	0.22	0	0	0

Relevant nodes for regression at $t = 1$

Path	t=0	t=1	S^2	Payoff t=1	PV(continuation value)	Payoff t=2	Conditional Payoff t=3
1	1	1.09	1.1881	0.01	0	0	0
4	1	0.93	0.8649	0.17	0.122429389	0.13	0
6	1	0.76	0.5776	0.34	0.310782296	0.33	0
7	1	0.92	0.8464	0.18	0.244858779	0.26	0
8	1	0.88	0.7744	0.22	0	0	0

The regression (in Excel, we use the linest() command) yields:

S^2	S	Intercept
1.356	**−3.335**	**2.038**
4.907	9.134	4.213
0.489011	0.142586	#N/A

Now, we can calculate fitted values and compare them to the value of immediate exercise:

Path	t=1	Payoff t=1	PV(continuation value)	Fitted continuation value	Max(cont. value, payoff t=1)
1	1.09	0.01	0	0.013485105	0.013485105
4	0.93	0.17	0.122429389	0.108749281	0.17
6	0.76	0.34	0.310782296	0.286064681	0.34
7	0.92	0.18	0.244858779	0.117009268	0.18
8	0.88	0.22	0	0.152762129	0.22

We wait in node 1, but we early exercise in nodes 4, 6, 7, and 8.

Question 19.16.

We can write down the optimal exercise schedule based on our results in exercise 19.15.

Path	t=1	t=2	t=3
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

The cash flows associated with this exercise schedule are:

Path	t=1	t=2	t=3
1	0	0	0
2	0	0	0
3	0	0	0.07
4	0.17	0	0
5	0	0	0
6	0.34	0	0
7	0.18	0	0
8	0.22	0	0

At $t = 0$, we therefore have the following continuation values

pv (continuation)
0
0
0.058468915
0.160099971
0
0.320199941
0.169517616
0.207188197

At $t = 0$, we can just estimate a regression with a constant, because the stock price and the squared stock price are constant and equal to one. All paths have an immediate exercise value of $1.1–1 = 0.1$, and thus need to be included in the regression. The coefficient on the constant of this regression is 0.114. This is larger than the intrinsic value of 0.1, therefore early exercise in never optimal in $t = 0$.

Hence, we can find the value of the American put option to be:

0	
0	
0.058468915	
0.160099971	
0	
0.320199941	
0.169517616	
0.207188197	
0.11443433	**= average(values above)**

For the European put option, we take the payoffs at $t = 3$, discount them at the risk-free rate, and then average the payoffs. We have:

Discounted Payoff	Payoff t=3
0	0
0	0
0.058468915	0.07
0.150348638	0.18
0	0
0.167054042	0.2
0.075174319	0.09
0	0
0.056380739	**= average()**

Question 19.17.

See Figure Three for a typical histogram. The Monte Carlo value, which is likely to be between 11.27 and 11.41, is higher than the Black-Scholes value of 10.91.