Chapter 20
Brownian Motion and Itô's Lemma

Question 20.1.

If $y = \ln(S)$ then $S = e^y$ and $dy = \left(\frac{\alpha(S,t)}{S} - \frac{\sigma(S,t)^2}{2S^2}\right) dt + \frac{\sigma(S,t)}{S} dZ_t$,

a) $dy = \left(\frac{\alpha}{e^y} - \frac{\sigma^2}{2e^{2y}}\right) dt + \frac{\sigma}{e^y} dZ_t.$

b) $dy = \left(\frac{\lambda a}{e^y} - \lambda - \frac{\sigma^2}{2e^{2y}}\right) dt + \frac{\sigma}{e^y} dZ_t.$

c) $dy = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t.$

Question 20.2.

If $y = S^2$ then $S = \sqrt{y}$ and $dy = \left(2S\alpha(S,t) + \sigma(S,t)^2\right) dt + 2S\sigma(S,t)\, dZ_t$ where $\alpha(S,t)$ is the drift of S and $\sigma(S,t)$ is the volatility of S. For the three specifications:

a) $dy = \left(2\alpha\sqrt{y} + \sigma^2\right) dt + 2\sqrt{y}\sigma dZ_t.$

b)

$$dy = \left(2\sqrt{y}\lambda\left(a - \sqrt{y}\right) + \sigma^2\right) dt + 2\sqrt{y}\sigma dZ_t \quad (1)$$

$$= \left(2\lambda a\sqrt{y} - 2\lambda y + \sigma^2\right) dt + 2\sqrt{y}\sigma dZ_t. \quad (2)$$

c) $dy = \left(2\alpha + \sigma^2\right) y dt + 2\sigma y dZ_t.$

Question 20.3.

If $y = 1/S$ then $S = 1/y$ and $dy = \left(-S^{-2}\alpha(S,t) + S^{-3}\sigma(S,t)^2\right) dt - S^{-2}\sigma(S,t)\, dZ_t$,

a) $dy = \left(-\alpha y^2 + \sigma^2 y^3\right) dt - \sigma y^2 dZ_t.$

b) $dy = \left(-\lambda\left(ay^2 - y\right) + \sigma^2 y^3\right) dt - \sigma y^2 dZ_t.$

c) $dy = \left(-\alpha + \sigma^2\right) y dt - \sigma y dZ_t.$

Question 20.4.

If $y = \sqrt{S}$ then $S = y^2$ and

$$dy = \left(\frac{1}{2}S^{-1/2}\alpha(S,t) - \frac{1}{8}S^{-3/2}\sigma(S,t)^2\right)dt + \frac{1}{2}S^{-1/2}\sigma(S,t)\,dZ_t \tag{3}$$

$$= \left(\frac{1}{2y}\alpha(S,t) - \frac{1}{8y^3}\sigma(S,t)^2\right)dt + \frac{1}{2y}\sigma(S,t)\,dZ_t \tag{4}$$

a) $dy = \left(\frac{\alpha}{2y} - \frac{\sigma^2}{8y^3}\right)dt + \frac{\sigma}{2y}dZ_t.$

b) $dy = \left(\frac{\lambda a}{2y} - \frac{\lambda}{2y^2} - \frac{\sigma^2}{8y^3}\right)dt + \frac{\sigma}{2y}dZ_t.$

c) $dy = \left(\frac{\alpha}{2} - \frac{\sigma^2}{8}\right)ydt + \frac{\sigma}{2}ydZ_t.$

Question 20.5.

Let $y = S^2Q^{0.5}$, then

$$\frac{dy}{y} = \left(2(\alpha_S - \delta_S) + \frac{\alpha_Q - \delta_Q}{2} + \sigma_S^2 - \frac{\sigma_Q^2}{8} + \rho\sigma_S\sigma_Q\right)dt + 2\sigma_S dZ_S + b\sigma_Q dZ_Q. \tag{5}$$

Question 20.6.

If $y = \ln(SQ) = \ln(S) + \ln(Q)$ then

$$dy = d\ln(S) + d\ln(Q) \tag{6}$$

$$= \left(\alpha_S - \delta_S - \sigma_S^2/2 + \alpha_Q - \delta_Q - \sigma_Q^2/2\right)dt + \sigma_S dZ_S + \sigma_Q dZ_Q. \tag{7}$$

Question 20.7.

With $\delta = 0$, the prepaid forward price for S_1^a is

$$F_{0,1}^P(S^a) = S_0^a \exp\left((a-1)r + \frac{1}{2}a(a-1)\sigma^2\right). \tag{8}$$

a) If $a = 2$, $F_{0,1}^P(S^2) = 100^2\exp(.06 + .4^2) = 12461.$

b) If $a = .5$, $F_{0,1}^P(S^{.5}) = 10\exp\left(-.03 - \frac{.4^2}{8}\right) = 9.5123.$

c) If $a = -2$, $F_{0,1}^P(S^{-2}) = 100^{-2}\exp\left(-.18 + 3(.4^2)\right) = 1.3499 \times 10^{-4}.$

Question 20.8.

Since the process $y = S^a Q^b$ follows geometric Brownian motion, i.e. $dy = \alpha_y y dt + \sigma_y y dZ_y$ the price of the claims will be $e^{-r} E^* (y_1) = y_0 e^{(\alpha_y - r)}$. We use Ito's lemma, as in equation (20.38), with $\delta = 0$ and $\alpha_S = \alpha_Q = r$ to arrive at the drift

$$\alpha_y = ar + br + \frac{1}{2} a (a-1) \sigma_S^2 + \frac{1}{2} b (b-1) \sigma_Q^2 + ab\rho\sigma_S\sigma_Q \tag{9}$$

$$= .06 (a+b) + \frac{.4^2}{2} a (a-1) + \frac{.2^2}{2} b (b-1) - .3 (.4) (.2) ab. \tag{10}$$

a) Since $a = b = 1$, $y_0 = 10000$ and $\alpha_y = .12 - .024 = .096$ hence the claim is worth $10000 e^{.096 - .06} = 10366.56$.

b) Since $a = 1$ and $b = -1$, $y_0 = 1$ and $\alpha_y = .2^2 + .024 = .064$ hence the claim is worth $e^{.064 - .06} = 1.004$.

c) Since $a = 1/2$ and $b = 1/2$, $y_0 = 100$ and $\alpha_y = .029$ hence the claim is worth $100 e^{.029 - .06} = 96.948$.

d) Since $a = -1$ and $b = -1$, $y_0 = 1/10000$ and $\alpha_y = .056$ hence the claim is worth $\left(e^{.056 - .06}\right) / 10000 = 9.960\,1 \times 10^{-5}$.

e) Since $a = 2$ and $b = 1$, $y_0 = 1000000$ and $\alpha_y = .292$ hence the claim is worth $1000000 e^{.292 - .06} = 1.2612$ million.

Question 20.9.

It is obvious if $t = 0$ the proposed solution will be equal to X_0. It is helpful to rewrite the solution as

$$X_t = e^{-\lambda t} \left(X_0 + a \left(e^{\lambda t} - 1 \right) + \sigma \int_0^t e^{\lambda s} dZ_s \right) = e^{-\lambda t} Y_t \tag{11}$$

where $dY_t = a\lambda e^{\lambda t} dt + \sigma e^{\lambda t} dZ_t$. Since $e^{-\lambda t}$ is deterministic,

$$dX_t = \left(-\lambda e^{-\lambda t} dt \right) Y_t + e^{-\lambda t} dY_t = (a\lambda - \lambda X_t) dt + \sigma dZ_t. \tag{12}$$

Question 20.10.

Note that if $V(S)$ satisfies the given equation, then

$$E^* (dV) = \left[(r - \delta) S V_S + \frac{1}{2} \sigma^2 S^2 V_{SS} \right] dt = rV dt. \tag{13}$$

Since $V(S) = kS^{h_1}$ where a is constant, showing $y = S^a$satisfies $E^*(dy) = rydt$ when $a = h_1$is sufficient (i.e. the constant term is irrelevant). Using Ito's lemma,

$$E^*(dy) = aS^{a-1}(r-\delta)S + \frac{1}{2}a(a-1)S^{a-2}\sigma^2 S^2 \tag{14}$$

$$= \left(a(r-\delta)y + a(a-1)\frac{\sigma^2}{2}y\right)dt. \tag{15}$$

If $E^*(dy) = rydt$ then a must satisfy

$$a(r-\delta) + a(a-1)\frac{\sigma^2}{2} = r. \tag{16}$$

The two solutions are h_1 and h_2 as given (12.11) and (12.12) which one can verify directly.

Question 20.11.

As discussed in the hint, consider a strategy of 1 unit in Q, $-Q\eta_i/(S_i\sigma_i)$ for both $i = 1$ and 2. Let I_t be the amount of money in the risk free asset. The value of the portfolio is

$$V_t = Q_t(1 - \eta_1/\sigma_1 - \eta_2/\sigma_2) + I_t. \tag{17}$$

The expected change in the value is

$$dV_t = dQ_t - \frac{Q_t\eta_1}{S_{1t}\sigma_1}dS_{1t} - \frac{Q_t\eta_2}{S_{2t}\sigma_2}dS_{2t} + rI_t dt \tag{18}$$

$$= \left(\left(\alpha_Q - \alpha_1\frac{\eta_1}{\sigma_1} - \alpha_2\frac{\eta_2}{\sigma_2}\right)Q + rI\right)dt \tag{19}$$

$$+ (\eta_1 Q - \eta_1 Q)\,dZ_1 + (\eta_2 Q - \eta_2 Q)\,dZ_2 \tag{20}$$

$$= \left(\left(\alpha_Q - \alpha_1\frac{\eta_1}{\sigma_1} - \alpha_2\frac{\eta_2}{\sigma_2}\right)Q + rI\right)dt. \tag{21}$$

Since this portfolio requires zero investment and there is no risk, the drift and V_t must be zero. Hence

$$\left(\alpha_Q - \alpha_1\frac{\eta_1}{\sigma_1} - \alpha_2\frac{\eta_2}{\sigma_2}\right)Q + r\left(1 - \frac{\eta_1}{\sigma_1} - \frac{\eta_2}{\sigma_2}\right)Q = 0. \tag{22}$$

Rearranging

$$\alpha_Q - r = \frac{\eta_1}{\sigma_1}(\alpha_1 - r) + \frac{\eta_2}{\sigma_2}(\alpha_2 - r). \tag{23}$$

Question 20.12.

We must try to find a position in S and Q that eliminates risk. Let us buy one unit of S and let θ be the position in Q. Let I_t be our bond investment. We have $V_t = S_t + \theta_t Q_t + I_t$ with $V_0 = 0$. Since this strategy must be self financing,

$$dV = \left(\alpha_S S + \theta \alpha_Q Q + rI\right) dt + (\sigma_S S - \eta\theta Q)\, dZ \tag{24}$$

hence we will set $\theta = \sigma_S S / (\eta Q)$. This will make our zero cost, self financing strategy riskless. Hence the drift and the value must be zero. Mathematically, if $V_t = 0$ then $I = -S - \frac{\sigma_S S}{\eta Q} Q$. The drift being zero implies

$$\alpha_S S + \frac{\sigma_S S}{\eta Q}\alpha_Q Q - r\left(S + \frac{\sigma_S S}{\eta}\right) = 0. \tag{25}$$

Dividing both sides by S and simplifying leads to

$$\alpha_Q = r - \frac{\alpha_S - r}{\sigma_S}\eta. \tag{26}$$

Since Q is negatively related to Z, if S has a positive risk premium then Q will negative risk premium.

Question 20.13.

In the following we define $y_t = S_t^a Q_t^b$.

a) From equation (20.38), the (real world) expected value of y_T is $E(y_T) = y_0 e^{mT}$ where

$$m = a(\alpha_S - \delta_S) + b\left(\alpha_Q - \delta_Q\right) + \frac{a(a-1)\sigma_S^2}{2} + \frac{b(b-1)\sigma_Q^2}{2} + ab\rho\sigma_S\sigma_Q \tag{27}$$

is the real world capital gain. Given a (real world) expected return α, the value of the claim is $e^{-\alpha T} E(y_T) = y_0 e^{(m-\alpha)T}$. Using Ito's lemma and problem 20.11, $\eta_1 = a\sigma_S$ and $\eta_2 = b\sigma_Q$. We then have

$$\alpha = r + a(\alpha_S - r) + b\left(\alpha_Q - r\right) \tag{28}$$

and the value of the claim being

$$y_0 e^{(m-\alpha)T} = S_0 Q_0 e^{-rT} e^{hT} \tag{29}$$

where $h = a(r - \delta_S) + b\left(r - \delta_Q\right) + \frac{1}{2}a(a-1)\sigma_S^2 + \frac{1}{2}b(b-1)\sigma_Q^2 + ab\rho\sigma_S\sigma_Q$. Note this agrees with $e^{-rT}E^*(y_T)$.

b) The expected return of y is α and the actual expected capital gain is m. The lease rate of y would have to be the difference $\delta^* = \alpha - m$ which equals

$$\delta^* = r(1-a-b) + a\delta_S + b\delta_Q - \frac{1}{2}a(a-1)\sigma_S^2 - \frac{1}{2}b(b-1)\sigma_Q^2 - ab\rho\sigma_S\sigma_Q.$$

The prepaid forward price must be $y_0 e^{-\delta^* T} = y_0 e^{(m-\alpha)T}$ which agrees with our previous answer. We can rewrite it in an informative way. The forward price for a security paying S^a is

$$F_{0,T}(S^a) = S^a e^{\left(a(r-\delta_S)+\frac{1}{2}a(a-1)\sigma_S^2\right)T}.$$

The forward price for Q^b is

$$F_{0,T}(Q^b) = Q^b e^{\left(b(r-\delta_Q)+\frac{1}{2}b(b-1)\sigma_Q^2\right)T}.$$

Thus, we can rewrite the prepaid forward price as

$$F_{0,T}^P(S^a Q^b) = e^{-rT} F_{0,T}(S^a) F_{0,T}(Q^b) e^{ab\rho\sigma_S\sigma_Q T}. \tag{31}$$

The expression on the right is the product of the forward prices times a factor that accounts for the covariance between the two assets. The discount factor converts it into a prepaid forward price.

Question 20.14.

As mentioned in the problem, σdZ appears in both dS and dQ. One can think of dQ as an alternative model for the stock (with dS being the standard geometric Brownian motion).

a) If there were no jumps, dQ would also be geometric Brownian motion. Since it has the same risk component, σdZ, α_Q must equal α. If we thought of Q as another traded asset, this naturally follows from no arbitrage.

b) If $Y_1 > 1$ then there are only positive jumps. We would therefore expect $\alpha_Q < \alpha$ to compensate for this. Mathematically, $dQ/Q - dS/S = \left(\alpha_Q - \alpha\right) dt + dq_1$. If a jump occurs, $dq_1 = Y_1 - 1 > 0$; if $\alpha_Q \geq \alpha$ we could buy Q and short S. The only risk we have is jump risk but this will always be "good" news for our portfolio. In order to avoid this arbitrage α_Q must be less than α.

If we use a weaker assumption $k_1 = E(Y_1 - 1) > 0$ and we assume the returns to S and Q should be the same (this makes sense if we are looking at Q as an alternative model instead of another stock) then we arrive at a similar result. The expected return to Q is $\alpha_Q + \lambda_1 k_1$; setting this equal to α implies $\alpha - \alpha_Q = \lambda_1 k_1 > 0$.

c) Let α^* be the expected return of Q. Note that α_Q is not the expected return, it is the expected return conditional on no jumps occurring. We have the following relationship,

$$\alpha^* = E\left(\frac{dQ}{Q}\right)/dt = \alpha_Q + k_1\lambda_1 + k_2\lambda_2 \tag{32}$$

where $k_i = E(Y_i - 1)$. Hence $\alpha_Q = \alpha^* - k_1\lambda_1 - k_2\lambda_2$. If $\alpha^* = \alpha$ (i.e. Q and S have the same expected return) then $\alpha - \alpha_Q = k_1\lambda_1 + k_2\lambda_2$. The sign of which could be positive or negative if there are no restriction on k_1 and k_2.