Chapter 21
The Black-Scholes Equation

Question 21.1.

If $V(S,t) = e^{-r(T-t)}$ then the partial derivatives are $V_S = V_{SS} = 0$ and $V_t = rV$. Hence $V_t + (r-\delta) SV_S + S^2\sigma^2 V_{SS}/2 = rV$.

Question 21.2.

If $V(S,t) = AS^a e^{\gamma t}$ then $V_t = \gamma V$, $V_S = aS^{a-1}e^{\gamma t} = aV/S$, and $V_{SS} = a(a-1)S^{a-2}e^{\gamma t} = a(a-1)V/S^2$. Therefore the left hand side of the Black-Scholes equation (21.11) is

$$V_t + (r-\delta) V_S S + V_{SS} S^2 \sigma^2/2 - rV = \left(\gamma - r + (r-\delta) a + \frac{\sigma^2}{2} a(a-1)\right) V. \tag{1}$$

We can rewrite the coefficient of V as

$$\gamma + (r-\delta) a + \frac{\sigma^2}{2} a(a-1) = \frac{\sigma^2}{2} a^2 + \left(r - \delta - \frac{\sigma^2}{2}\right) a + \gamma - r. \tag{2}$$

From the quadratic formula, this has roots

$$a = \frac{-\left(r-\delta-\frac{\sigma^2}{2}\right)}{\sigma^2} \pm \frac{\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 - 4\frac{\sigma^2}{2}(\gamma - r)}}{\sigma^2}. \tag{3}$$

Simplifying,

$$a = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r-\gamma)}{\sigma^2}}. \tag{4}$$

Note, for a given γ, these are the only values for a that will satisfy the PDE.

Question 21.3.

If $V(S,t) = e^{-r(T-t)} S^a \exp\left(\left(a(r-\delta) + \frac{1}{2}a(a-1)\sigma^2\right)(T-t)\right)$, we have $V(S,T) = S_T^a$, hence the boundary condition is satisfied. Note that V is of the form $AS^a e^{\gamma t}$, where $\gamma = r - a(r-\delta) - \frac{1}{2}a(a-1)\sigma^2$. The previous problem's result shows γ must solve

$$a = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(r-\gamma)}{\sigma^2}}. \tag{5}$$

Letting $k = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)$, we have to check

$$a \overset{?}{=} k \pm \sqrt{k^2 + \frac{2(r-\gamma)}{\sigma^2}}. \tag{6}$$

This is equivalent to checking

$$k^2 + \frac{2(r-\gamma)}{\sigma^2} \overset{?}{=} (a-k)^2. \tag{7}$$

Expanding, this becomes

$$\frac{2(r-\gamma)}{\sigma^2} \overset{?}{=} a^2 - 2a\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right). \tag{8}$$

Solving for γ,

$$\gamma \overset{?}{=} r - \frac{\sigma^2 a^2}{2} + a\left(\frac{\sigma^2}{2} - (r-\delta)\right) = r - a(r-\delta) - \frac{\sigma^2}{2}a(a-1) \tag{9}$$

which is confirmed. One could also do this as a partial derivative exercise.

Question 21.4.

Defining $V(S,t) = Ke^{-r(T-t)} + Se^{-\delta(T-t)}$ we have $V_t = rKe^{-r(T-t)} + \delta Se^{-\delta(T-t)}$, $V_S = e^{-\delta(T-t)}$ and $V_{SS} = 0$. The Black-Scholes equation is satisfied for $V_t + (r-\delta)V_S S + V_{SS}S^2\sigma^2/2$ is

$$rKe^{-r(T-t)} + \delta Se^{-\delta(T-t)} + (r-\delta)e^{-\delta(T-t)}S \tag{10}$$

$$= r\left(Ke^{-r(T-t)} + Se^{-\delta(T-t)}\right) = rV. \tag{11}$$

This also follows from the result that linear combinations of solutions of the PDE are also solutions. The boundary condition is $V(S,T) = K + S_T$, i.e. we receive one share and K dollars. Similarly, a long forward contract with value $Se^{-\delta(T-t)} - Ke^{-r(T-t)}$ will solve the PDE.

Question 21.5.

Let $V = Se^{-\delta(T-t)}N(d_1)$. Note that d_1 depends on both S and t. We have

$$V_S = e^{-\delta(T-t)}\left(N(d_1) + S\frac{\partial N(d_1)}{\partial S}\right) = e^{-\delta(T-t)}\left(N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}}\right) \tag{12}$$

hence

$$(r-\delta)\,SV_S = (r-\delta)\,V + \frac{(r-\delta)}{\sigma\sqrt{T-t}}e^{-\delta(T-t)}SN'(d_1)\,. \tag{13}$$

Similarly,

$$V_{SS} = e^{-\delta(T-t)}\left(\frac{N'(d_1)}{S\sigma\sqrt{T-t}} + \frac{N''(d_1)}{S\sigma^2(T-t)}\right) = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}\left(1-\frac{d_1}{\sigma\sqrt{T-t}}\right) \tag{14}$$

where we used the fact $N''(x) = -xN'(x)$. We have

$$\frac{S^2\sigma^2 V_{SS}}{2} = \frac{\sigma Se^{-\delta(T-t)}N'(d_1)}{2\sqrt{T-t}}\left(1-\frac{d_1}{\sigma\sqrt{T-t}}\right). \tag{15}$$

The partial with respect to t is

$$\begin{aligned} V_t &= \delta V + Se^{-\delta(T-t)}N'(d_1)\left(\frac{\ln(S/K)}{2\sigma(T-t)^{3/2}} - \frac{r-\delta+\sigma^2/2}{2\sigma(T-t)^{1/2}}\right) \\ &= \delta V + \frac{Se^{-\delta(T-t)}N'(d_1)}{2(T-t)}\left(d_1 - 2\frac{\left(r-\delta+\sigma^2/2\right)}{\sigma}\sqrt{T-t}\right). \end{aligned} \tag{16}$$

Adding equations (13), (15), and (16), all terms cancel except the rV term from equation (13), hence $V_t + (r-\delta)\,SV_S + S^2\sigma^2 V_{SS}/2 = rV$ which was to be shown.

Question 21.6.

Let $V(S,t) = e^{-r(T-t)}N(d_2)$; we must show V solves the PDE $V_t + (r-\delta)\,SV_S + S^2\sigma^2V_{SS}/2 = rV$. Note that

$$d_2 = \frac{\ln(S/K)}{\sigma\sqrt{T-t}} + \left(\frac{r-\delta-\sigma^2/2}{\sigma}\right)\sqrt{T-t} \tag{17}$$

depends on both S and t. Beginning with the first term in the PDE,

$$\begin{aligned} V_t &= rV + e^{-r(T-t)}N'(d_2)\left(\frac{\ln(S/K)}{2\sigma(T-t)^{3/2}} - \frac{r-\delta-\sigma^2/2}{2\sigma(T-t)^{1/2}}\right) \\ &= rV + \frac{e^{-r(T-t)}N'(d_2)}{2(T-t)}\left(d_2 - \frac{2\left(r-\delta-\sigma^2/2\right)}{\sigma}\sqrt{T-t}\right). \end{aligned} \tag{18}$$

Since $V_S = e^{-r(T-t)}N'(d_2)/\left(S\sigma\sqrt{T-t}\right)$ the second term in the PDE is

$$(r-\delta)\,SV_S = \left(\frac{r-\delta}{\sigma\sqrt{T-t}}\right)e^{-r(T-t)}N'(d_2)\,. \tag{19}$$

The second partial of V with respect to S is

$$V_{SS} = \frac{e^{-r(T-t)}\left(N''(d_2) - N'(d_2)\right)}{S^2\sigma^2(T-t)} = -\frac{e^{-r(T-t)}N'(d_2)}{S^2\sigma^2(T-t)}\left(d_2 + \sigma\sqrt{T-t}\right) \tag{20}$$

where we use the property $N''(x) = -xN'(x)$. The third term in the PDE is therefore

$$\frac{S^2\sigma^2V_{SS}}{2} = -\frac{e^{-r(T-t)}N'(d_2)}{2(T-t)}\left(d_2 + \sigma\sqrt{T-t}\right). \tag{21}$$

Adding equations (18), (19), and (21), all terms cancel expect the rV term in equation (18); i.e. V satisfies the PDE.

Question 21.7.

The two preceding problems, show that each term in the Black-Scholes call option formula satisfies the PDE (these are all or nothing options); since linear combination of solutions to PDEs are also solutions, the Black-Scholes formula solves the PDE. That is

$$V(S,t) = \underbrace{Se^{-\delta(T-t)}N(d_1)}_{\text{Problem 21.5}} - \underbrace{Ke^{-r(T-t)}N(d_2)}_{K\times\text{ Problem 21.6}} \tag{22}$$

The only thing left is to show the boundary condition, $V(S,T) = \max(S-K, 0)$. The first term is $SN(d_1)$. As in the text's discussion of the European call option, at $t = T$,

$$N(d_1) = N(d_2) = \begin{cases} 1 & \text{if } S > K \\ 0 & \text{if } S < K \end{cases} \tag{23}$$

hence $V(S,T) = S - K$ if $S \geq K$ and $V(S,T) = 0$ otherwise.

Question 21.8.

These bets are all or nothing options. The cash bets being worth, per dollar, $e^{-rT}N(d_2)$ if we receive \$1 if $S_T > K$ and $e^{-rT}N(-d_2)$ if we receive \$1 if $S_T < K$. The stock bets being worth, per share, $SN(d_1)$ if we receive 1 share if $S_T > K$ and $SN(-d_1)$ if we receive 1 share if $S_T < K$. (Note we are assuming the current time is $t = 0$ and the bet is for the stock price T years from now).

a) By setting $K = Se^{(r-\delta)T}, d_2 = -\sigma\sqrt{T}/2$ the value of the bet that the share price will exceed the forward price is $e^{-rT}N(-\sigma\sqrt{T}/2)$. This is always less than the opposite bet, which has value $e^{-rT}N(\sigma\sqrt{T}/2)$.

b) If denominated in cash, we could make the bet fair by setting the strike price equal to $K = Se^{(r-\delta-.5\sigma^2)T}$, which is the median (50% of the probability is above this value). This will make $d_2 = 0$ and the bets worth $e^{-rT}/2$ which is not a surprise since the sum of the two bets must be worth e^{-rT}. Using $T = 1, r = 6\%, \sigma = 30\%$, we have $K = 100e^{.06-.3^2/2} = 101.51$.

c) If denominated in shares, we could make the bet fair by setting the strike price equal to $K = Se^{(r-\delta+.5\sigma^2)T} = 100e^{.06+.3^2/2} = 111.07$, which is above the forward price. This makes $d_1 = 0$ and the bets worth $S/2 = 50$.

Question 21.9.

Let $S = 100$ and $K = 106.184$ which is the forward price. The first bet is worth $V_1 = SN(\sigma\sqrt{T}/2) - e^{-rT}KN(-\sigma\sqrt{T}/2)$ and the second bet is worth $V_2 = KN(\sigma\sqrt{T}/2) - SN(-\sigma\sqrt{T}/2)$. The difference in the values

$$V_1 - V_2 = \left(S - Ke^{-rT}\right)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) + N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right) = S - Ke^{-rT}. \tag{24}$$

Since K is the forward price, $K = Se^{rT}$ which implies $V_1 = V_2$. This is simply put call parity; if the strike price is the forward price, $C - P$ must equal the value of an obligation to buy the asset for the forward price which, by definition is zero. Using the parameters, $\sigma = 30\%, r = 6\%$, and $T = 1$, both bets should be worth \$11.92.

Question 21.10.

If we purchase one unit of the claim, $-V_S$ shares, and invest W in the risk free bond, our investment is worth $I = V(S,t) - V_S S + W = 0$. By purchasing one claim, we will receive a dividend of Γdt that will be added to dI. The change in the investment value is

$$dI = \Gamma dt + V_t dt + V_S dS + \frac{\sigma^2 S^2 V_{SS} dt}{2} - V_S dS - V_S \delta S dt + rW dt \tag{25}$$

$$= \left(\Gamma + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - V_S \delta S + rW\right) dt. \tag{26}$$

Since this is risk free and is (initially) a zero investment, both the drift and I must be zero. This implies $W = V_S S - V$ and

$$\Gamma + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - V_S \delta S + r(V_S S - V) = 0, \tag{27}$$

hence

$$\Gamma + V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta) V_S S = rV. \tag{28}$$

Note that if we assume Γ is a continuous yield of the claim (rather than a \$ per unit rate), the first term would be ΓV rather than Γ.

Question 21.11.

Using the notation from Proposition 21.1, $\eta = .02 + 2\,(.2)\,.3\,(.5) = .08, \delta^* = .06 - 2\,(.06 - .01) - .5^2 = -.29$. The function V is the prepaid forward price of S, $S_0 e^{-\eta T}$. The value of the claim is

$$90^2 e^{(.06+.29)2} 50 e^{-.08(2)} = 694,983. \tag{29}$$

Using proposition 20.4, the value should be

$$S_0 e^{-\delta T} \left(Q_0^b e^{\left(b\left(r-\delta_Q\right)+.5b(b-1)\sigma_Q^2\right)2} \right) e^{b\rho\sigma\sigma_Q T} \tag{30}$$

which equals

$$50e^{-.04} \left(90^2 e^{\left(.1+.5^2\right)2}\right) e^{-.12} = 694,983. \tag{31}$$

Question 21.12.

Setting $b = -1$ and using Proposition 21.1, we change the dividend yield of S to $\eta = .02 - .2\,(.3)\,(.5) = -.01$. The prepaid forward price, i.e. V in equation (21.35), is $S_0 e^{-\eta T}$. Letting $\delta^* = .06 + (.06 - .01) - .5^2 = -.14$, we have the value of the claim being

$$\frac{1}{90} e^{.2(2)} \left(50 e^{.01(2)}\right) = 0.8455. \tag{32}$$

Using Proposition 20.4, the claim should be worth

$$S_0 e^{-\delta T} \left(Q_0^b e^{\left(b\left(r-\delta_Q\right)+.5b(b-1)\sigma_Q^2\right)2} \right) e^{b\rho\sigma\sigma_Q T} \tag{33}$$

which equals

$$50e^{-.04} \left(90^{-1} e^{\left(-.05+.5^2\right)2}\right) e^{.03(2)} = 0.8455. \tag{34}$$

Note that Proposition 20.4 derives the forward price; upon discounting, the forward price of S becomes $S_0 e^{-\delta T}$ and the forward price of Q^b terms does not get discounted.

Question 21.13.

Let $P(Q, S, 0)$ be the current ($t = 0$) no arbitrage value of the claim that pays $\left[Q_T - F_{0,T}\right] \times$ $\max(0, S_T - K)$. Since $F_{0,T}(Q) = Qe^{(r-\delta_Q)T}$ (a "known" number)

$$P(Q, S, 0) = Qe^{(r-\delta_Q)T} V(S, K, \sigma_S, r, T, \delta - \rho\sigma\sigma_S) - Qe^{(r-\delta_Q)T} V(S, K, \sigma_S, r, T, \delta). \quad (35)$$

where $V(\cdot)$ is the Black-Scholes call option formula; note that there is a different dividend yield in the two equations. We immediately see that, since $\rho < 0$, the first option will be worth less than the second and we shouldn't accept this offer. Intuitively, since $\ln(S)$ and $\ln(Q)$ are negatively correlated, when $Q_T > F_{0,T}(Q)$, the call option is more likely to be out of the money. Using $K = 50$, the claim will be worth

$$90e^{(.06-.01)2}(7.98 - 10.39) = -239.71. \quad (36)$$

Question 21.14.

Using Proposition 21.1, since $b = 1$, the insurance payoff should be worth

$$Qe^{(r-\delta_Q)T} V(S, K, \sigma_S, r, T, \delta - \rho\sigma\sigma_S) \quad (37)$$

hence we should use a dividend yield of $.02 + .2(.3)(.5) = .05$ making the put relatively more valuable. For $K = 50$, $V = 7.09$ hence the insurance is worth $90e^{(.06-.01)2}(7.09) = 705.21$. If we wanted to insure $90e^{(.06-.01)2} = 99.465$ units, it would cost $90e^{(.06-.01)2}(6.05) = 601.77$. This is intuitive since $\ln(S)$ and $\ln(Q)$ are negatively correlated. When Q is high, S is more likely to be low making the insurance payout larger (the holder has the right to sell *more* units for K).