Chapter 24

Interest Rate Models

Question 24.1.

a) $F = P(0,2)/P(0,1) = .8495/.9259 = .91749.$

b) Using Black's Formula,

$$BSCall\,(.8495, .9009 \times .9259, .1, 0, 1, 0) = \$0.0418. \tag{1}$$

c) Using put call parity for futures options,

$$p = c + KP(0,1) - FP(0,1) = .0418 + .8341 - .8495 = \$0.0264 \tag{2}$$

d) Since $1 + K_R = 1.1$, the caplet is worth 1.1 one year put options on the two year bond with strike price $1/1.1 = 0.9009$ which is the same strike as before. Hence the caplet is worth $1.11 \times .0264 = .0293$.

Question 24.2.

a) The two year forward price is $F = P(0,3)/P(0,2) = .7722/.8495 = .90901$.

b) Since $FP(0,2) = P(0,3)$ the first input into the formula will be .7722. The present value of the strike price is $.9P(0,2) = .9 \times .8495 = .76455$. We can use this as the strike with no interest rate; we could also use a strike of .9 with an interest rate equal to the 2 year yield. Either way the option is worth

$$BSCall\,(.7722, .76455, .105, 0, 2, 0) = \$0.0494 \tag{3}$$

c) Using put call parity for futures options,

$$p = c + KP(0,2) - FP(0,2) = .0494 + .76455 - .7722 = \$0.4175. \tag{4}$$

d) The caplet is worth 1.11 two year put options with strike $1/1.11 = .9009$. The no interest formula will use $(.9009)(.8495) = .7653$ as the strike. The caplet has a value of

$$1.11BSPut\,(.7722, .7653, .105, 0, 2, 0) = \$0.0468. \tag{5}$$

Question 24.3.

We must sum three caplets. The one year option has a value of .0248, the two year option has a value .0404, and the three year option has a value of .0483. The three caplets have a combined value of

$$1.115\,(.0248 + .0404 + .0483) = \$0.1266. \tag{6}$$

Question 24.4.

A flat yield curve implies the two bond prices are $P_1 = e^{-.08(3)} = .78663$ and $P_2 = e^{-.08(6)} = .61878$. If we have purchased the three year bond, the duration hedge is a position of

$$N = -\frac{1}{2}\frac{P_1}{P_2} = -\frac{1}{2}e^{3(.08)} = -.63562 \tag{7}$$

in the six year bond. Notice the total cost of this strategy is

$$V_{8\%} = .78663 - .63562\,(.61878) = .39332 \tag{8}$$

which implies we will owe $.39332e^{.08/365} = .39341$ in one day. If yields rise to 8.25%, our portfolio will have a value

$$V_{8.25\%} = e^{-.0825(3-1/365)} - .63562e^{-.0825(6-1/365)} = .39338. \tag{9}$$

If yields fall to 7.75%, the value will be

$$V_{7.75\%} = e^{-.0775(3-1/365)} - .63562e^{-.0775(6-1/365)} = .39338. \tag{10}$$

Either way we lose .00003. This is a binomial version of the impossibility of a no arbitrage flat (stochastic) yield curve.

Question 24.5.

For this question, let P_1 be the price of the 4 year, 5% coupon bond and let P_2 be the price of the 8 year, 7% coupon bond. Note we use a continuous yield and assume the coupon rates simple. We also use a continuous yield version of duration, i.e. $D = -\frac{\partial P/\partial y}{P}$.

a) The bond prices are

$$P_1 = \sum_{i=1}^{4} .05e^{-.06(i)} + e^{-.06(4)} = .95916. \tag{11}$$

and

$$P_2 = \sum_{i=1}^{8} .07e^{-.06(i)} + e^{-.06(8)} = 1.0503. \quad (12)$$

The (modified) durations are

$$D_1 = \frac{\sum_{i=1}^{4} .05ie^{-.06(i)} + 4e^{-.06(4)}}{.95916} = 3.7167. \quad (13)$$

and

$$D_2 = \frac{\sum_{i=1}^{8} .07ie^{-.06(i)} + 8e^{-.06(8)}}{1.0503} = 6.4332. \quad (14)$$

b) If we buy one 4-year bond the duration hedge involves a position of

$$N = -\frac{D_1}{D_2}\frac{P_1}{P_2} = -.5276 \quad (15)$$

of the 8-year bond. This has a total cost of $.9516 - .5276\,(1.0503) = .39746$. The next day we will owe $.39746e^{.06/365} = .39753$. If yields rise in the next instant to 6.25 then bond prices will be

$$P_1 = \sum_{i=1}^{4} .05e^{-.0625(i-1/365)} + e^{-.0625(4-1/365)} = .95045 \quad (16)$$

and

$$P_2 = \sum_{i=1}^{8} .07e^{-.0625(i-1/365)} + e^{-.0625(8-1/365)} = 1.0338. \quad (17)$$

The duration hedge will have a value of $.95045 - .5276\,(1.0338) = .40502 < .39753$ and we will profit. If the yields fall to 5.75 then one can check $P_1 = .96827$ and $P_2 = 1.0675$. The duration hedge will have a value of $.96827 - .5276\,(1.0675) = .40506$ and we will profit again.

Question 24.6.

Note that the interest rate risk premium of zero implies $\phi = 0$.

a) Beginning with the CIR model and using equation (23.37),

$$\gamma = \sqrt{a^2 + 2\sigma^2} = \sqrt{.2^2 + 2\,(.44721)^2} = .66332. \quad (18)$$

Let A_2 and B_2 be the 2 year bond's A and B term in equation (23.37). Then

$$A_2 = \left(\frac{2\gamma e^{.2+\gamma}}{(.2+\gamma)\left(e^{2\gamma}-1\right)+2\gamma} \right)^{.04/.44721^2} = .96718 \tag{19}$$

and

$$B_2 = \frac{2\left(e^{2\gamma}-1\right)}{(.2+\gamma)\left(e^{2\gamma}-1\right)+2\gamma} = 1.4897. \tag{20}$$

This gives a price of the two year bond equal to

$$P(0,2) = .96718e^{-1.4897(.05)} = .89776. \tag{21}$$

The delta is $P_r = -B_2 P(0,2) = -1.4897(.89776) = -1.3374$ and the gamma is $P_{rr} = B_2^2 P(0,2) = 1.4897^2(.89776) = 1.9923$. Similar analysis for the ten year bond will yield a price of $P(0,10) = .6107$, a delta of -1.4119, and a gamma of 3.2643. The "true" duration of the bonds should be $-P_r/P$ which equal 1.49 and 2.31 (respectively) quite different from 2 and 10 years. The "true" convexity should be P_{rr}/P which equals 2.22 and 5.35; the traditional convexities are $P_{yy}/P = 4$ and 100.
Using similar notation for the Vasicek model and equation (23.24) the two year bond price is derived from the components

$$\bar{r} = .1 - 0.5\left(.1^2\right)/.2^2 = -.025, \tag{22}$$

$$B_2 = \left(1 - e^{-2(.2)}\right)/.2 = 1.6484, \tag{23}$$

and

$$A_2 = e^{-.025(1.6484-2)-.16484^2/.8} = .97514. \tag{24}$$

The two year bond will be worth $P(0,2) = .97514e^{-1.6484(.05)} = .89799$. As in the CIR analysis, the delta will be $P_r = -B_2 P(0,2) = -1.6484(.89799) = -1.4802$ and a gamma of $P_{rr} = B_2^2 P(0,2) = 1.6484^2(.89799) = 2.44$. Similarly, the price of the 10 year bond is .735, the delta is -3.1776, and the gamma is 13.74. The "true" durations $-P_r/P$ are 1.65 and 4.32 are substantially different from 2 and 10. The convexity measures P_{rr}/P which equal 2.72 and 18.694 are also quite different from 4 and 100.

b) The duration hedge will use a position of

$$N_{duration} = -\frac{2}{10}\frac{P(0,2)}{P(0,10)} = -.2(.89799)/.735 = -.24435. \tag{25}$$

The delta hedge is

$$N_{delta} = -\frac{P_r\,(0,2)}{P_r\,(0,10)} = -\frac{1.4802}{3.177} = -.4658. \tag{26}$$

The duration hedged portfolio has a cost of $.89799 - .24435\,(.735) = .71839$ and the delta hedge costs $.89799 - .4658\,(.735) = .55563$. The one day standard deviation for r will be $.05 \pm .1/\sqrt{365}$. In the "up" scenario the bond prices will become $P_2 = .8904$ and $P_{10} = .7186$. The return in the up scenario for the two hedges are

$$return_{duration} = .8904 - .24435\,(.7186) - .71839e^{.05/365} = -.00368 \tag{27}$$

and

$$return_{delta} = .8904 - .4658\,(.7186) - .55563e^{.05/365} = -.00003. \tag{28}$$

In the "down" scenario the bond prices will be $P_2 = .905895$ and $P_{10} = .751818$. The return in the down scenario for the two hedges are

$$return_{duration} = .905895 - .24435\,(.751818) - .71839e^{.05/365} = .0037 \tag{29}$$

and

$$return_{delta} = .905895 - .4658\,(.751818) - .55563e^{.05/365} = -.00009. \tag{30}$$

The delta hedge error is significantly smaller (in absolute terms) in both scenarios.

c) The one day standard deviation for the CIR model is $\sigma_{CIR}\sqrt{r/365} = 5.234\,2 \times 10^{-3}$ which is, by design, the same as part b). The duration hedge is

$$N_{duration} = -\frac{2}{10}\frac{P\,(0,2)}{P\,(0,10)} = -.2\frac{.89776}{.6107} = -.29401. \tag{31}$$

which has a total cost of $.89776 - .29401\,(.6107) = .71821$. The delta hedge is

$$N_{delta} = -\frac{P_r\,(0,2)}{P_r\,(0,10)} = -\frac{1.3374}{1.4119} = -.94723. \tag{32}$$

which has a total cost of $.89776 - .94723\,(.6107) = .3193$. If r rises by the one day standard deviation, the bond prices will be $P_2 = .89092$ and $P_{10} = .603436$. This leads to "up" returns of

$$return_{duration} = .89092 - .29401\,(.6034) - .71821e^{.05/365} = -.0048 \tag{33}$$

and

$$return_{delta} = .89092 - .94723\,(.6034) - .3193e^{.05/365} = -6.42 \times 10^{-6}. \tag{34}$$

If the short term rate falls by the one day standard deviation, the bond prices will be $P_2 = .9049$ and $P_{10} = .6182$, leading to "down" returns of

$$return_{duration} = .9049 - .2940\,(.6182) - .71821e^{.05/365} = .0048. \tag{35}$$

and

$$return_{delta} = .9049 - .94723\,(.6182) - .3193e^{.05/365} = -2.13 \times 10^{-5}. \tag{36}$$

Without rounding errors the return is closer to -6×10^{-6}.

Question 24.7.

See Table One for the binomial tree of the short rate. The one-year bond has a price of $P\,(0,1) = e^{-.10} = .9048$. The two year bond has a price of

$$P\,(0,2) = e^{-.10}\left(.5e^{-.12} + .5e^{-.08}\right) = .8189 \tag{37}$$

which has a yield of $-\ln(.8189)/2 = .0999$. The three year bond price is

$$P\,(0,3) = e^{-.10}\left[.5e^{-.12}\left(.5e^{-.14} + .5e^{-.1}\right) + .5e^{-.08}\left(.5e^{-.1} + .5e^{-.06}\right)\right] = .7416 \tag{38}$$

which has a yield of $-\ln(.7416)/3 = .0997$. For the four year bond price we look at the value it has next year in the up state

$$P_u = e^{-.12}\left[.5e^{-.14}\left(.5e^{-.16} + .5e^{-.12}\right) + .5e^{-.1}\left(.5e^{-.12} + .5e^{-.08}\right)\right] = .6984 \tag{39}$$

and in the down state

$$P_d = e^{-.08}\left[.5e^{-.1}\left(.5e^{-.12} + .5e^{-.08}\right) + .5e^{-.06}\left(.5e^{-.08} + .5e^{-.04}\right)\right] = .7874. \tag{40}$$

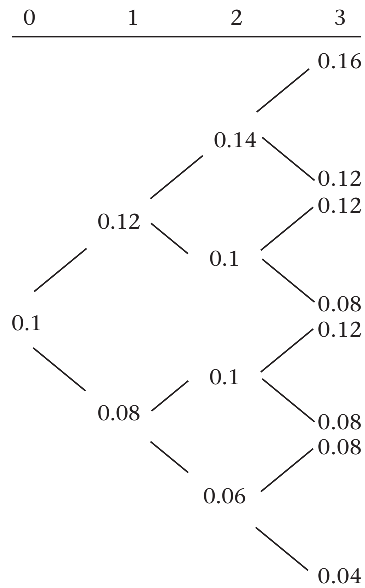
The four year bond has a value

$$P\,(0,4) = e^{-.1}\,(.5\,(.6984) + .5\,(.7874)) = .6722 \tag{41}$$

which has a yield of $-\ln(.6722)/4 = .0993$. As in Example 23.3, yields decline with maturity (more of a "Jensen's inequality effect") and are less than the expected future short rate of 10%.

However, the short rate moving up or down 2% (instead of 4%) dampens this effect considerably with yields on the three bonds being very close to 10%.

Table One (Problem 24.7)



Question 24.8.

Instead of one long equation we will work backwards. In year 3, the four year bond is worth the same value as the 1-year bond in the terminal nodes of Figure 24.6. In year two the bond will be worth three possible values, $.8321\left(\frac{.8331+.8644}{2}\right) = .70624$, $.8798\left(\frac{.8644+.8906}{2}\right) = .77202$, and $.9153\frac{(.8906+.9123)}{2} = .8251$. In year one, the bond will be worth two possible values, $.8832\left(\frac{.70624+.77202}{2}\right) = .6528$ or $.9023\left(\frac{.77202+.8251}{2}\right) = .72054$. Finally, the current value is the discounted expected value

$$P(0,4) = .9091\left(\frac{.6528 + .72054}{2}\right) = .6243. \tag{42}$$

Question 24.9.

Next year, the bond prices will be .6528 or .72054 which imply yields of $.6528^{-1/3} - 1 = .15276$ or $.72054^{-1/3} - 1 = .11544$. The yield volatility is then

$$0.5 \times \ln\left(\frac{.15276}{.11544}\right) = 14\%. \tag{43}$$

Question 24.10.

The value of the year-2 cap payment has been shown to be $V_2 = 1.958$. We must add to this the value of the year-3 cap payment and the value of the year-1 cap payment. In year 2, the year-3 cap payment will be worth three possible values: $.8321\left(\frac{6.689+3.184}{2}\right) = 4.1077$, $.8798\left(\frac{3.184+.25}{2}\right) = 1.5106$, or $.9153\,(.125) = .11441$. In year 1, the year-3 cap payment will be worth two possible values: $.8832\left(\frac{4.1077+1.5106}{2}\right) = 2.481$ or $.9023\left(\frac{1.5106+.11441}{2}\right) = .73312$. Hence the year-3 cap payment has a current value of

$$V_3 = .9091\left(\frac{2.481 + .73312}{2}\right) = 1.461. \tag{44}$$

The year-1 cap payment has a value of $V_1 = .9091\left(\frac{1.078}{2}\right) = .49$. Summing the three we have

$$V_1 + V_2 + V_3 = .49 + 1.958 + 1.461 = 3.909. \tag{45}$$

Question 24.11.

Note there will be minor discrepancies due to rounding errors.

From Table 24.2, $P\,(0,3)\,/P\,(0,4) - 1 = .7118/.6243 - 1 = .14016$

In year 4 we receive $r\,(3,4) - \bar{r}_A$. The value $\bar{r}_A$ will be worth $\bar{r}_A B\,(0,4) = .6243\bar{r}_A$. In year 3 (using a notional amount of 100) the $r\,(3,4)$ is worth four possible values, $.8331\,(20.03) = 16.687$, $.8644\,(15.68) = 13.554$, $.8906\,(12.28) = 10.937$, and $.9123\,(9.62) = 8.7763$. Working backwards, the three possible year 2 values are $.8321\left(\frac{16.687+13.554}{2}\right) = 12.582$, $.8798\left(\frac{13.554+10.937}{2}\right) = 10.774$, and $.9153\left(\frac{10.937+8.7763}{2}\right) = 9.0218$. The two possible year 1 values are $.8832\left(\frac{12.582+10.774}{2}\right) = 10.314$ and $.9023\left(\frac{10.774+9.0218}{2}\right) = 8.9309$. This gives a present value of $100r\,(3,4)$ equal to $.9091\left(\frac{10.314+8.9309}{2}\right) = 8.7478$. In order for the contract to have zero current value we require

$$\bar{r}_A \times 62.43 = 8.7478 \Longrightarrow \bar{r}_A = 14.012\%. \tag{46}$$

Question 24.12.

See Table Two on the next page for the bond prices which are the same for the two trees. The one year bonds are simply $1/\,(1+r)$ where r is the short rate from the given trees. For the two year bonds we can solve recursively with formulas such as $B\,(0,2) = B\,(0,1) \times \left[\frac{B(0,1)_u + B(0,1)_d}{2}\right]$ where $B\,(0,1)$ is the node's 1 year bond and $B\,(0,1)_u$ and $B\,(0,1)_d$ are the one year bond prices at the next node. Once we have two year bonds, three year bond values can be given by $B\,(0,3) = B\,(0,1) \times \left[\frac{B(0,2)_u + B(0,2)_d}{2}\right]$ and similarly for the four and 5 year bonds.

Table Two (Problem 24.12)

Tree #1					Tree #2				
0.08	0.07676	0.0817	0.07943	0.07552	0.08	0.08112	0.08749	0.08261	0.07284
	0.10362	0.10635	0.09953	0.09084		0.09908	0.10689	0.10096	0.08907
		0.13843	0.12473	0.10927			0.1306	0.12338	0.10891
			0.1563	0.13143				0.15078	0.13317
				0.15809					0.16283

One Year Bond Prices

0.925926	0.928712	0.924471	0.926415	0.929783	**0.925926**	0.924967	0.919549	0.923694	0.932105
	0.906109	0.903873	0.90948	0.916725		0.909852	0.903432	0.908298	0.918215
		0.878403	0.889102	0.901494			0.884486	0.890171	0.901786
			0.864827	0.883837				0.868976	0.88248
				0.863491					0.859971

Two Year Bond Prices

0.849454	0.849002	0.848615	0.855316	**0.849453**	0.843098	0.842303	0.854564
	0.807468	0.812845	0.826816		0.81337	0.812397	0.826552
		0.770328	0.793671			0.77797	0.794151
			0.755569				0.757074

Three Year Bond Prices

0.766885	0.771509	0.777541	**0.766884**	0.765271	0.772934
	0.717264	0.732357		0.7235	0.732097
		0.680428			0.686018

Four Year Bond Prices

0.689247	0.70113	**0.689246**	0.696052
	0.640069		0.645138

Five Year Bond Price

0.620926	**0.620921**

Question 24.13.

See Table Three which uses the prices from Table Two to first determine next year's up and down yields on the current 2, 3, and 4 year bonds. The volatility is then $.5 \times \ln(y_u/y_d)$. We see that Tree #2 has yield volatilities of 10% for all three bonds; whereas, Tree #1 has higher yield volatilities for all three bonds of 15%, 14% and 13% (respectively). Although Tree #2 has lower one year volatilities for the three bonds, there may be areas of the tree where it has yield higher volatilities.

Table Three (Problem 24.13)

Tree #1

	Next Year's Yields		
	2 year	3 year	4 year
up	0.07676	0.085289	0.090318
down	0.10362	0.112852	0.117139
Volatilities	0.150023	0.140013	0.130013

Tree #2

	Next Year's Yields		
	2 year	3 year	4 year
up	0.08112	0.089083	0.093272
down	0.09908	0.108807	0.11392
Volatilities	0.099999	0.100003	0.099988

For example, in the three year 8.261% state, the two year bond has a one year yield volatility of $.5 \times \ln(.09084/.07552) = 9.24\%$ in Tree #1 and a one year yield volatility of $.5 \times \ln(.08907/.07284) = 10.06\%$.

Question 24.14.

See Table Four for the numerical answers to parts a) and b). Let $r_f(i)$ and $r_e(i)$ be the one period forward rate for borrowing at time i.

Table Four (Problem 24.14)

1 year forward rate	American	European	Difference
Year 2	9.002%	9.019%	0.017%
Year 3	10.767%	10.803%	0.036%
Year 4	11.264%	11.308%	0.044%
Year 5	11.003%	11.041%	0.037%

1 year forward rate	American	European	Difference
Year 2	9.003%	9.003%	0.000%
Year 3	10.767%	10.788%	0.021%
Year 4	11.264%	11.299%	0.035%
Year 5	11.004%	11.048%	0.044%

Year 3 European Calculations

0.0917689	0.087322
	0.110899
0.10803279	

Year 3 European Calculations

0.0916379	0.089898
	0.10804
0.10787872	

Year 4 European Calculations

0.08671898	0.085475	0.082722
	0.101838	0.101351
0.11307958		0.123429

Year 4 European Calculations

0.08664816	0.085901	0.084401
	0.101259	0.101338
0.11298736		0.121245

Year 5 European Calculations

0.07609656	0.077625	0.077682	0.077059
	0.086744	0.089484	0.090998
0.11040541		0.101981	0.107003
			0.125192

Year 5 European Calculations

0.07614931	0.076312	0.07572	0.074778
	0.08817	0.089286	0.089912
0.11048209		0.104526	0.107746
			0.128608

a) Note the forward rates only depend on the initial bond prices; for example, $r_f(2) = B(0,1)/B(0,2) - 1 = .925926/.849454 - 1 = 9.0025\%$. This immediately implies the yield volatilities do not affect these forward rates.

b) These rates were computed by formulas such as

$$r_e(2) = \frac{r_u + r_d}{2} \tag{47}$$

and

$$r_e(3) = \frac{1}{B(0,2)}\left(\frac{1}{2(1+r_d)}\frac{r_{dd}+r_{du}}{2} + \frac{1}{2(1+r_u)}\frac{r_{du}+r_{uu}}{2}\right). \tag{48}$$

c) From Table Four, we see that the difference between the two settlement styles is larger for the high volatility tree (#1) for the year 2 and 3 forward rates. In addition, the difference is larger for the later years. In looking at the short rate trees we see that the short rate tree #1 has a lower dispersion in year 4 (ranging from 7.55% to 15.81%) than it does in tree #2 (ranging from 7.28% to 16.28%). This causes the difference for the 5 year rates to be more pronounced for tree #2.

Question 24.15.

At each node (with short rate r) the cap pays $\frac{250000}{1+r} \times \max(r - .105, 0)$. Table Five gives the payoffs as well as the valuation of the cap. For tree #1, the cap it worth 5612.60 and for tree #2 it is worth 5236.66. The higher initial volatility of tree #1 outweighs the lower volatility in year 4.

Table Five (Problem 24.15)

Tree #1

0.08	0.07676	0.0817	0.07943	0.07552
	0.10362	0.10635	0.09953	0.09084
		0.13843	0.12473	0.10927
			0.1563	0.13143
				0.15809

Payments

0	0	0	0	0
	0	305.0572	0	0
		7341.251	4385.497	962.3446
			11091.41	5839.955
				11460.68

Total Valuation

5612.59552	1882.371	202.2818	0	0
	10240.84	3851.441	437.6163	0
		18752.54	7409.467	962.3446
			18572.44	5839.955
				11460.68

Tree #2

0.08	0.08112	0.08749	0.08261	0.07284
	0.09908	0.10689	0.10096	0.08907
		0.1306	0.12338	0.10891
			0.15078	0.13317
				0.16283

Payments

0	0	0	0	0
	0	426.8717	0	0
		5660.711	4090.335	881.4962
			9945.428	6214.866
				12433.03

Total Valuation

5236.65511	1880.54	184.0618	0	0
	9430.635	3882.117	400.3307	0
		16847.93	7248.822	881.4962
			18047.71	6214.866
				12433.03