Chapter 9
Parity and Other Option Relationships

Question 9.1.

This problem requires the application of put-call-parity. We have:

$$\begin{aligned} & P\,(35, 0.5) = C\,(35, 0.5) - e^{-\delta T} S_0 + e^{-rT} 35 \\ \Leftrightarrow\ & P\,(35, 0.5) = \$2.27 - e^{-0.06\times 0.5} 32 + e^{-0.04\times 0.5} 35 = \$5.523. \end{aligned}$$

Question 9.2.

This problem requires the application of put-call-parity. We have:

$$\begin{aligned} & S_0 - C\,(30, 0.5) + P\,(30, 0.5) - e^{-rT} 30 = PV\,(dividends) \\ \Leftrightarrow\ & PV\,(dividends) = 32 - 4.29 + 2.64 - 29.406 = \$0.944. \end{aligned}$$

Question 9.3.

a) We can calculate an initial investment of:

$$-800 + 75 - 45 = -770.$$

This position yields \$815 after one year for sure, because either the sold call commitment or the bought put cancel out the stock price. Therefore, we have a one-year rate of return of 0.05844, which is equivalent to a continuously compounded rate of: 0.05680.

b) We have a risk-less position in a) that pays more than the risk-free rate. Therefore, we should borrow money at 5%, and buy a large amount of the aggregate position of a), yielding a sure return of .68%.

c) An initial investment of \$775.252 would yield \$815 after one year, invested at the risk-less rate of return of 5%. Therefore, the difference between call and put prices should be equal to \$24.748.

d) Following the same argument as in c), we obtain the following results:

Strike Price	Call-Put
780	58.04105
800	39.01646
820	19.99187
840	0.967283

Question 9.4.

We can make use of the put-call-parity for currency options:

$$
\begin{aligned}
& +P(K,T) = -e^{-r_f T} x_0 + C(K,T) + e^{-rT} K \\
\Leftrightarrow \quad & P(K,T) = -e^{-0.04} 0.95 + 0.0571 + e^{-0.06} .93 = -0.91275 + 0.0571 + 0.87584 = 0.0202.
\end{aligned}
$$

A \$0.93 strike European put option has a value of \$0.0202.

Question 9.5.

The payoff of the one-year yen-denominated put on the euro is max[0, 100 − x_1Y/E], where x_1 is the future uncertain Y/E exchange rate.

The payoff of the corresponding one-year Euro-denominated call on yen is max[0, $1/x_1 - 1/100$]. Its premium is, using equation (9.7),

$$
\begin{aligned}
& P_Y(x_0, K, T) = x_0 K C_E\left(\frac{1}{x_0}, \frac{1}{K}, T\right) \\
\Leftrightarrow \quad & C_E\left(\frac{1}{x_0}, \frac{1}{K}, T\right) = \frac{P_Y(x_0, K, T)}{x_0 K} \\
\Leftrightarrow \quad & C_E\left(\frac{1}{95}, \frac{1}{100}, 1\right) = \frac{8.763}{9500} = 0.00092242
\end{aligned}
$$

Question 9.6.

a) We can use put-call-parity to determine the forward price:

$$
\begin{aligned}
& + C(K,T) - P(K,T) = PV(forward\ price) - PV(strike) = e^{-rT} F_{0,T} - K e^{-rT} \\
\Leftrightarrow \quad & F_{0,T} = e^{rT}\left[+C(K,T) - P(K,T) + K e^{-rT}\right] \\
& \qquad = e^{0.05*0.5}\left[\$0.0404 - \$0.0141 + \$0.9 e^{-0.05*0.5}\right] \\
\Leftrightarrow \quad & F_{0,T} = \$0.92697.
\end{aligned}
$$

b) Given the forward price from above and the pricing formula for the forward price, we can find the current spot rate:

$$
\begin{aligned}
& F_{0,T} = x_0 e^{(r-r_f)T} \\
\Leftrightarrow \quad & x_0 = F_{0,T} e^{-(r-r_f)T} = \$0.92697 e^{-(0.05-0.035)0.5} = \$0.92.
\end{aligned}
$$

Question 9.7.

a) We make use of the version of the put-call-parity that can be applied to currency options. We have:

$$
\begin{aligned}
&+ P(K,T) = -e^{-r_f T} x_0 + C(K,T) + e^{-rT} K \\
\Leftrightarrow \quad & P(K,T) = -e^{-0.01} 0.009 + 0.0006 + e^{-0.05}.009 = -0.00891045 + 0.0006 + 0.00856106 \\
& \qquad = 0.00025.
\end{aligned}
$$

b) The observed option price is too high. Therefore, we sell the put option and synthetically create a long put option, perfectly offsetting the risks involved. We have:

Transaction	$t = 0$	$t = T, x < K$	$t = T, x > K$
Sell Put	0.0004	$-(K - x) = x - K$	0
Buy Call	-0.0006	0	$x - K$
Sell e^{-r_f} Spot	$0.009e^{-0.001} = 0.00891$	$-x$	$-x$
Lend PV(strike)	-0.00856	K	K
Total	0.00015	0	0

We have thus demonstrated the arbitrage opportunity.

c) We can use formula (9.7.) to determine the value of the corresponding Yen-denominated at-the-money put, and then use put-call-parity to figure out the price of the Yen-denominated at-the-money call option.

$$
\begin{aligned}
& C_\$ (x_0, K, T) = x_0 K P_Y \left(\frac{1}{x_0}, \frac{1}{K}, T \right) \\
\Leftrightarrow \quad & P_Y \left(\frac{1}{x_0}, \frac{1}{K}, T \right) = \frac{C_\$ (x_0, K, T)}{x_0 K} \\
\Leftrightarrow \quad & P_Y \left(\frac{1}{0.009}, \frac{1}{0.009}, T \right) = \frac{0.0006}{(0.009)^2} = 7.4074Y
\end{aligned}
$$

Now, we can use put-call-parity, carefully handling the interest rates (since we are now making transactions in Yen, the \$-interest rate is the foreign rate):

$$
\begin{aligned}
& C_Y (K,T) = e^{-r_f T} x_0 + P_Y (K,T) - e^{-rT} K \\
\Leftrightarrow \quad & C\left(\frac{1}{0.009}, 1 \right) = e^{-0.05} \frac{1}{0.009} + 7.4074 - e^{-0.01} \frac{1}{0.009} = 105.692 + 7.4074 - 110.0055 \\
& \qquad = 3.093907.
\end{aligned}
$$

We have used the direct relationship between the yen-denominated dollar put and our answer to a) and the put-call-parity to find the answer.

Question 9.8.

Both equations (9.13) and (9.14) are violated. We use a call bull spread and a put bear spread to profit from these arbitrage opportunities.

		Expiration or Exercise		
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$S_T > 55$
Buy 50 strike call	-9	0	$S_T - 50$	$S_T - 50$
Sell 55 strike call	$+10$	0	0	$55 - S_T$
TOTAL	$+1$	0	$S_T - 50 > 0$	$5 > 0$

		Expiration or Exercise		
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$S_T > 55$
Buy 55 strike put	-6	$55 - S_T$	$55 - S_T$	0
Sell 50 strike put	7	$S_T - 50$	0	0
TOTAL	$+1$	$5 > 0$	$55 - S_T > 0$	0

Please note that we initially receive money, and that at expiration the profit is non-negative. We have found arbitrage opportunities.

Question 9.9.

Both equations (9.15) and (9.16) of the textbook are violated. We use a call bear spread and a put bull spread to profit from these arbitrage opportunities.

		Expiration or Exercise		
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$S_T > 55$
Buy 55 strike call	-10	0	0	$S_T - 55$
Sell 50 strike call	$+16$	0	$50 - S_T$	$50 - S_T$
TOTAL	$+6$	0	$50 - S_T > -5$	-5

		Expiration or Exercise		
Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$S_T > 55$
Buy 50 strike put	-7	$50 - S_T$	0	0
Sell 55 strike put	14	$S_T - 55$	$S_T - 55$	0
TOTAL	$+7$	-5	$S_T - 55 > -5$	0

Please note that we initially receive more money than our biggest possible exposure in the future. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

Question 9.10.

Both equations (9.17) and (9.18) of the textbook are violated. To see this, let us calculate the values. We have:

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} = \frac{18 - 14}{55 - 50} = 0.8 \quad \text{and} \quad \frac{C(K_2) - C(K_3)}{K_3 - K_2} = \frac{14 - 9.50}{60 - 55} = 0.9,$$

which violates equation (9.17) and

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} = \frac{10.75 - 7}{55 - 50} = 0.75 \quad \text{and} \quad \frac{P(K_3) - P(K_2)}{K_3 - K_2} = \frac{14.45 - 10.75}{60 - 55} = 0.74,$$

which violates equation (9.18).

We calculate lambda in order to know how many options to buy and sell when we construct the butterfly spread that exploits this form of mispricing. Because the strike prices are symmetric around 55, lambda is equal to 0.5.

Therefore, we use a call and put butterfly spread to profit from these arbitrage opportunities.

Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$55 \leq S_T \leq 60$	$S_T > 60$
Buy 1 50 strike call	-18	0	$S_T - 50$	$S_T - 50$	$S_T - 50$
Sell 2 55 strike calls	$+28$	0	0	$110 - 2 \times S_T$	$110 - 2 \times S_T$
Buy 1 60 strike call	-9.50	0	0	0	$S_T - 60$
TOTAL	$+0.50$	0	$S_T - 50 \geq 0$	$60 - S_T \geq 0$	0

Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$55 \leq S_T \leq 60$	$S_T > 60$
Buy 1 50 strike put	-7	$50 - S_T$	0	0	0
Sell 2 55 strike puts	21.50	$2 \times S_T - 110$	$2 \times S_T - 110$	0	0
Buy 1 60 strike put	-14.45	$60 - S_T$	$60 - S_T$	$60 - S_T$	0
TOTAL	$+0.05$	0	$S_T - 50 \geq 0$	$60 - S_T \geq 0$	0

Please note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

Question 9.11.

Both equations (9.17) and (9.18) of the textbook are violated. To see this, let us calculate the values. We have:

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} = \frac{22 - 9}{100 - 80} = 0.65 \quad \text{and} \quad \frac{C(K_2) - C(K_3)}{K_3 - K_2} = \frac{9 - 5}{105 - 100} = 0.8,$$

which violates equation (9.17) and

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} = \frac{21 - 4}{100 - 80} = 0.85 \qquad \text{and} \qquad \frac{P(K_3) - P(K_2)}{K_3 - K_2} = \frac{24.80 - 21}{105 - 100} = 0.76,$$

which violates equation (9.18).

We calculate lambda in order to know how many options to buy and sell when we construct the butterfly spread that exploits this form of mispricing. Using formula (9.19), we can calculate that lambda is equal to 0.2. To buy and sell round lots, we multiply all the option trades by 5.

We use an asymmetric call and put butterfly spread to profit from these arbitrage opportunities.

Transaction	$t = 0$	$S_T < 80$	$80 \leq S_T \leq 100$	$100 \leq S_T \leq 105$	$S_T > 105$
Buy 2 80 strike calls	-44	0	$2 \times S_T - 160$	$2 \times S_T - 160$	$2 \times S_T - 160$
Sell 10 100 strike calls	$+90$	0	0	$1000 - 10 \times S_T$	$1000 - 10 \times S_T$
Buy 8 105 strike calls	-40	0	0	0	$8 \times S_T - 840$
TOTAL	$+6$	0	$2 \times S_T - 160 > 0$	$840 - 8 \times S_T \geq 0$	0

Transaction	$t = 0$	$S_T < 80$	$80 \leq S_T \leq 100$	$100 \leq S_T \leq 105$	$S_T > 105$
Buy 2 80 strike puts	-8	$160 - 2 \times S_T$	0	0	0
Sell 10 100 strike puts	$+210$	$10 \times S_T - 1000$	$10 \times S_T - 1000$	0	0
Buy 8 105 strike puts	-198.4	$840 - 8 \times S_T$	$840 - 8 \times S_T$	$840 - 8 \times S_T$	0
TOTAL	$+3.6$	0	$2 \times S_T - 160 > 0$	$840 - 8 \times S_T \geq 0$	0

Please note that we initially receive money and have non-negative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

Question 9.12.

a) Equation (9.15) of the textbook is violated. We use a call bear spread to profit from this arbitrage opportunity.

		Expiration or Exercise		
Transaction	$t = 0$	$S_T < 90$	$90 \leq S_T \leq 95$	$S_T > 95$
Sell 90 strike call	$+10$	0	$90 - S_T$	$90 - S_T$
Buy 95 strike call	-4	0	0	$S_T - 95$
TOTAL	$+6$	0	$90 - S_T > -5$	-5

Please note that we initially receive more money than our biggest possible exposure in the future. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

b) Now, equation (9.15) is not violated anymore. However, we can still construct an arbitrage opportunity, given the information in the exercise. We continue to sell the 90-strike call and buy the 95-strike call, and we loan our initial positive net balance for two years until expiration. It is important that the options be European, because otherwise we would not be able to tell whether the 90-strike call could be exercised against us sometime (note that we do not have information regarding any dividends).

We have the following arbitrage table:

		Expiration $t = T$		
Transaction	$t = 0$	$S_T < 90$	$90 \le S_T \le 95$	$S_T > 95$
Sell 90 strike call	$+10$	0	$90 - S_T$	$90 - S_T$
Buy 95 strike call	-5.25	0	0	$S_T - 95$
Loan 4.75	-4.75	5.80	5.80	5.8
TOTAL	0	5.80	$95.8 - S_T > 0$	$+0.8$

In all possible future states, we have a strictly positive payoff. We have created something out of nothing—we demonstrated arbitrage.

c) We will first verify that equation (9.17) is violated. We have:

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} = \frac{15 - 10}{100 - 90} = 0.5 \quad \text{and} \quad \frac{C(K_2) - C(K_3)}{K_3 - K_2} = \frac{10 - 6}{105 - 100} = 0.8,$$

which violates equation (9.17).

We calculate lambda in order to know how many options to buy and sell when we construct the butterfly spread that exploits this form of mispricing. Using formula (9.19), we can calculate that lambda is equal to 1/3. To buy and sell round lots, we multiply all the option trades by 3.

We use an asymmetric call and put butterfly spread to profit from these arbitrage opportunities.

Transaction	$t = 0$	$S_T < 90$	$90 \le S_T \le 100$	$100 \le S_T \le 105$	$S_T > 105$
Buy 1 90 strike calls	-15	0	$S_T - 90$	$S_T - 90$	$S_T - 90$
Sell 3 100 strike calls	$+30$	0	0	$300 - 3 \times S_T$	$300 - 3 \times S_T$
Buy 2 105 strike calls	-12	0	0	0	$2 \times S_T - 210$
TOTAL	$+3$	0	$S_T - 90 \ge 0$	$210 - 2 \times S_T \ge 0$	0

We indeed have an arbitrage opportunity.

Question 9.13.

We have to think carefully about the benefits and costs associated with early exercise. For the American put option, the usual benefit of early exercise is by exercising we can earn interest on the strike. We lose this interest if we continue to hold the option unexercised. However, when the interest rate is zero, there is no interest benefit from early exercise. There are two benefits to deferring exercise: first, there is a volatility benefit from waiting. Second, if the stock pays dividends, the put is implicitly short the stock without having to pay dividends. By exercising the put is converted into an actual short position with an obligation to pay the dividends. So with a zero interest rate, we will never use the early exercise feature of the American put option, and the American put is equivalent to a European put.

For the American call option, dividends on the stock are the reason why we want to receive the stock earlier, and we benefit from waiting, because we can continue to earn interest on the strike. Now, the interest rate is zero, so we do not have this benefit associated with waiting to exercise. However, we saw that there is a second benefit to waiting: the insurance protection, which will not be affected by the zero interest rate. Finally, we will not exercise the option if it is out-of-the-money. Therefore, there may be circumstances in which we will early exercise, but we will not always early exercise.

Question 9.14.

This question is closely related to question 9.13. In this exercise, the strike is not cash anymore, but rather one share of Apple. In parts a) and b), there is no benefit in keeping Apple longer, because the dividend is zero.

a) The underlying asset is the stock of Apple, which does not pay a dividend. Therefore, we have an American call option on a non-dividend-paying stock. It is never optimal to early exercise such an option.

b) The underlying asset is the stock of Apple, and the strike consists of AOL. As AOL does not pay a dividend, the interest rate on AOL is zero. We will therefore never early exercise the put option, because we cannot receive earlier any benefits associated with holding Apple – there are none. If Apple is bankrupt, there is no loss from not early exercising, because the option is worth max[0, AOL – 0], which is equivalent to one share of AOL, because of the limited liability of stock. As AOL does not pay dividends, we are indifferent between holding the option and the stock.

c) For the American call option, dividends on the stock are the reason why we want to receive the stock earlier, and now Apple pays a dividend. We usually benefit from waiting, because we can continue to earn interest on the strike. However, in this case, the dividend on AOL remains zero, so we do not have this benefit associated with waiting to exercise. Finally, we saw that there is a second benefit to waiting: the insurance protection, which will not be affected by the zero AOL dividend. Therefore, there now may be circumstances in which we will early exercise, but we will not always early exercise.

For the American put option, there is no cost associated with waiting to exercise the option, because exercising gives us a share of AOL, which does not pay interest in form of a dividend. However, by early exercising we will forego the interest we could earn on Apple. Therefore, it is again never optimal to exercise the American put option early.

Question 9.15.

We demonstrate that these prices permit an arbitrage. We buy the cheap option, the longer lived one-and-a-half year call, and sell the expensive one, the one-year option.

Let us consider the two possible scenarios. First, let us assume that $S_t < 105.127$. Then, we have the following table:

			time $= T$	time $= T$
Transaction	$t = 0$	time $= t$	$S_T \leq 107.788$	$S_T > 107.788$
Sell 105.127 strike call	$+11.924$	0	—	—
Buy 107.788 strike call	-11.50	0	0	$S_T - 107.788$
TOTAL	0.424	0	0	$S_T - 107.788$

Clearly, there is no arbitrage opportunity. However, we will need to check the other possibility as well, namely that $S_t \geq 105.127$. This yields the following no-arbitrage table:

			time $= T$	time $= T$
Transaction	$t = 0$	time $= t$	$S_T \leq 107.788$	$S_T > 107.788$
Sell 105.127 strike call	$+11.924$	$105.127 - S_t$	—	—
Keep stock		$+S_t$	$-S_T$	$-S_T$
Loan 105.127 @ 5% for 0.5 years		-105.127	$+107.788$	$+107.788$
Buy 107.788 strike call	-11.50	0	0	$S_T - 107.788$
TOTAL	0.424	0	$107.788 - S_T \geq 0$	0

We have thus shown that in all states of the world, there exist an initial positive payoff and non-negative payoffs in the future. We have demonstrated the arbitrage opportunity.

Question 9.16.

Short call perspective:

		time $= T$	time $= T$
Transaction	$t = 0$	$S_T \leq K$	$S_T > K$
Sell call	$+C^B$	0	$K - S_T$
Buy put	$-P^A$	$K - S_T$	0
Buy share	$-S^A$	S_T	S_T
Borrow @ r_B	$+Ke^{-r_B T}$	$-K$	$-K$
TOTAL	$C^B - P^A - S^A + Ke^{-r_B T}$	0	0

In order to preclude arbitrage, we must have: $C^B - P^A - S^A + Ke^{-r_B T} \leq 0$.

Long call perspective:

		time $= T$	time $= T$
Transaction	$t = 0$	$S_T \leq K$	$S_T > K$
Buy call	$-C^A$	0	$S_T - K$
Sell put	$+P^B$	$S_T - K$	0
Sell share	$+S^B$	$-S_T$	$-S_T$
Lend @ r_L	$-Ke^{-r_L T}$	$+K$	$+K$
TOTAL	$-C^A + P^B + S^B - Ke^{-r_L T}$	0	0

In order to preclude arbitrage, we must have: $-C^A + P^B + S^B - Ke^{-r_L T} \leq 0$.

Question 9.17.

In this problem we consider whether parity is violated by any of the option prices in Table 9.1. Suppose that you buy at the ask and sell at the bid, and that your continuously compounded lending rate is 1.9% and your borrowing rate is 2%. Ignore transaction costs on the stock, for which the price is $84.85. Assume that IBM is expected to pay a $0.18 dividend on November 8 (prior to expiration of the November options). For each strike and expiration, what is the cost if you

a) Buy the call, sell the put, short the stock, and lend the present value of the strike price plus dividend?

		Calls		Puts		Expiry	Maturity (in years)	Value of position
75	November	9.9	10.3	0.2	0.25	11/20/04	0.0986	−0.2894
80	November	5.3	5.6	0.6	0.7	11/20/04	0.0986	−0.1800
85	November	1.9	2.1	2.1	2.3	11/20/04	0.0986	−0.1706
90	November	0.35	0.45	5.5	5.8	11/20/04	0.0986	−0.1113
75	January	10.5	10.9	0.7	0.8	1/22/2005	0.2712	−0.1443
80	January	6.5	6.7	1.45	1.6	1/22/2005	0.2712	−0.1686
85	January	3.2	3.4	3.1	3.3	1/22/2005	0.2712	−0.1929
90	January	1.2	1.35	6.1	6.3	1/22/2005	0.2712	−0.1172

All costs are negative. The positions yield a payoff of zero in the future, independent of the future stock price. Therefore, the given prices preclude arbitrage.

b)

		Calls		Puts		Expiry	Maturity (in years)	Value of position
75	November	9.9	10.3	0.2	0.25	11/20/04	0.0986	−0.1680
80	November	5.3	5.6	0.6	0.7	11/20/04	0.0986	−0.2279
85	November	1.9	2.1	2.1	2.3	11/20/04	0.0986	−0.2377
90	November	0.35	0.45	5.5	5.8	11/20/04	0.0986	−0.2976
75	January	10.5	10.9	0.7	0.8	1/22/2005	0.2712	−0.3760
80	January	6.5	6.7	1.45	1.6	1/22/2005	0.2712	−0.2030
85	January	3.2	3.4	3.1	3.3	1/22/2005	0.2712	−0.2301
90	January	1.2	1.35	6.1	6.3	1/22/2005	0.2712	−0.2571

Again, all costs are negative. The positions yield a payoff of zero in the future, independent of the future stock price. Therefore, the given prices preclude arbitrage.

Question 9.18.

Consider the January 80, 85, and 90 call option prices in Table 9.1.

a) Does convexity hold if you buy a butterfly spread, buying at the ask price and selling at the bid?

Since the strike prices are symmetric, lambda is equal to 0.5. Therefore, to buy a long butterfly spread, we buy the 80-strike call, sell two 85 strike calls and buy one 90 strike call.

$$\frac{C\left(K_1\right)-C\left(K_2\right)}{K_2-K_1}=\frac{6.70-3.20}{85-80}=0.7 \quad \text{and} \quad \frac{C\left(K_2\right)-C\left(K_3\right)}{K_3-K_2}=\frac{3.20-1.35}{90-85}=0.37$$

Convexity holds.

b)

$$\frac{C\left(K_1\right)-C\left(K_2\right)}{K_2-K_1}=\frac{6.5-3.4}{85-80}=0.62 \quad \text{and} \quad \frac{C\left(K_2\right)-C\left(K_3\right)}{K_3-K_2}=\frac{3.4-1.20}{90-85}=0.44$$

Convexity holds.

c) Does convexity hold if you are a market-maker either buying or selling a butterfly, paying the bid and receiving the ask?

A market maker can buy a butterfly spread at the prices we sell it for. Therefore, the above convexity conditions are the ones relevant for market makers. Convexity is not violated from a market maker's perspective.