The Binomial Tree and Lognormality

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- The usefulness of the binomial pricing model hinges on the binomial tree providing a reasonable representation of the stock price distribution
- The binomial tree approximates a lognormal distribution

The Random Walk Model

- It is often said that stock prices follow a random walk
- Imagine that we flip a coin repeatedly
  - Let the random variable Y denote the outcome of the flip
  - If the coin lands displaying a head, $Y = 1$; otherwise, $Y = -1$
  - If the probability of a head is 1/2, we say the coin is fair
  - After n flips, with the i^{th} flip denoted Y_i, the cumulative total, Z_n, is

$$Z_n = \sum_{i=1}^{n} Y_i \qquad (11.8)$$

The Random Walk Model (cont'd)

- We can represent the process followed by Z_n in term of the *change* in Z_n

$$Z_n - Z_{n-1} = Y_n$$

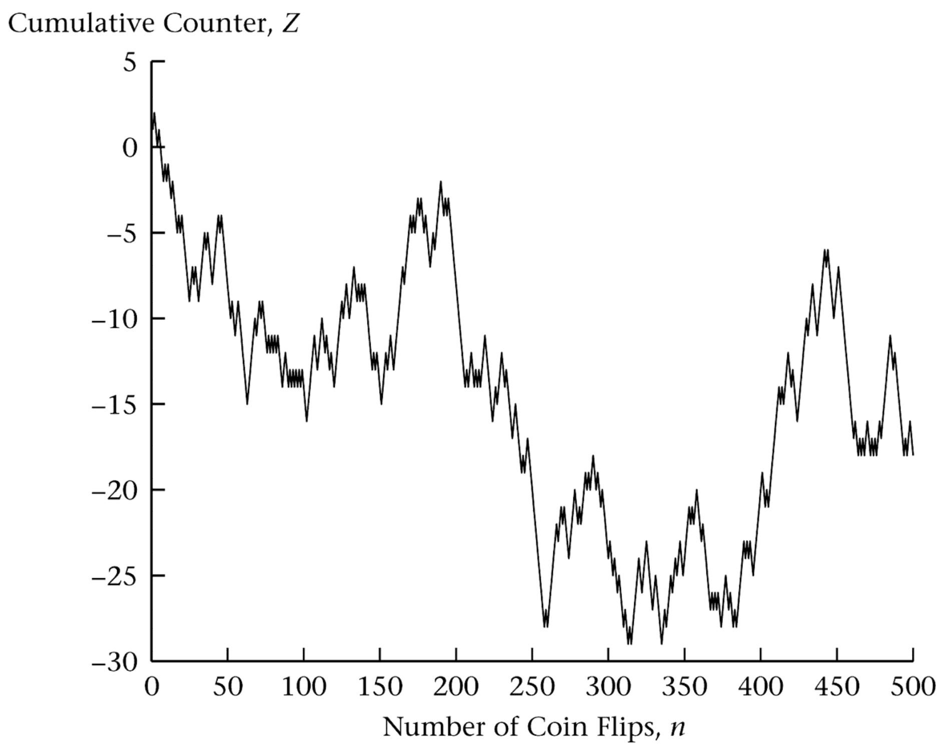
or

$$\text{Heads: } Z_n - Z_{n-1} = +1$$

$$\text{Tails: } Z_n - Z_{n-1} = -1$$

The Random Walk Model (cont'd)

- A random walk, where with heads, the change in *Z* is 1, and with tails, the change in *Z* is – 1



The Random Walk Model (cont'd)

- The idea that asset prices should follow a random walk was articulated in Samuelson (1965)
- In efficient markets, an asset price should reflect all available information. In response to new information the price is equally likely to move up or down, as with the coin flip
- The price after a period of time is the initial price plus the cumulative up and down movements due to new information

Modeling Stock Prices As a Random Walk

- The above description of a random walk is not a satisfactory description of stock price movements. There are at least three problems with this model
  - If by chance we get enough cumulative down movements, the stock price will become negative
  - The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price
  - The stock, on average, should have a positive return. However, the random walk model taken literally does not permit this
- The binomial model is a variant of the random walk model that solves all of these problems

Continuously Compounded Returns

- The binomial model assumes that continuously compounded returns are a random walk
- Some important properties of continuously compounded returns
  - The logarithmic function computes returns from prices
  - The exponential function computes prices from returns
  - Continuously compounded returns are additive
  - Continuously compounded returns can be less than –100%

The Standard Deviation of Returns

- Suppose the continuously compounded return over month *i* is $r_{\text{monthly},i}$. The annual return is

$$r_{annual} = \sum_{i=1}^{12} r_{monthly,i}$$

- The variance of the annual return is (11.14)

$$Var(r_{annual}) = Var\left(\sum_{i=1}^{12} r_{monthly,i}\right)$$

The Standard Deviation of Returns (cont'd)

- Suppose that returns are uncorrelated over time and that each month has the same variance of returns. Then from equation (11.14) we have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

where σ^2 is the annual variance

- The annual standard deviation is

$$\sigma_{monthly} = \frac{\sigma}{\sqrt{12}}$$

- If we split the year into *n* periods of length *h* (so that *h* = 1/*n*), the standard deviation over the period of length *h* is

$$\sigma_h = \sigma\sqrt{h} \qquad (11.15)$$

The Binomial Model

- The binomial model is

$$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{h}}$$

- Taking logs, we obtain

$$ln(S_{t+h} / S_t) = (r - \delta)h \pm \sigma\sqrt{h} \qquad (11.16)$$

  – Since *ln* (S_{t+h}/S_t) is the continuously compounded return from t to $t+h$, the binomial model is simply a particular way to model the continuously compounded return
  – That return has two parts, one of which is certain, $(r–\delta)h$, and the other of which is uncertain, $\sigma\sqrt{h}$

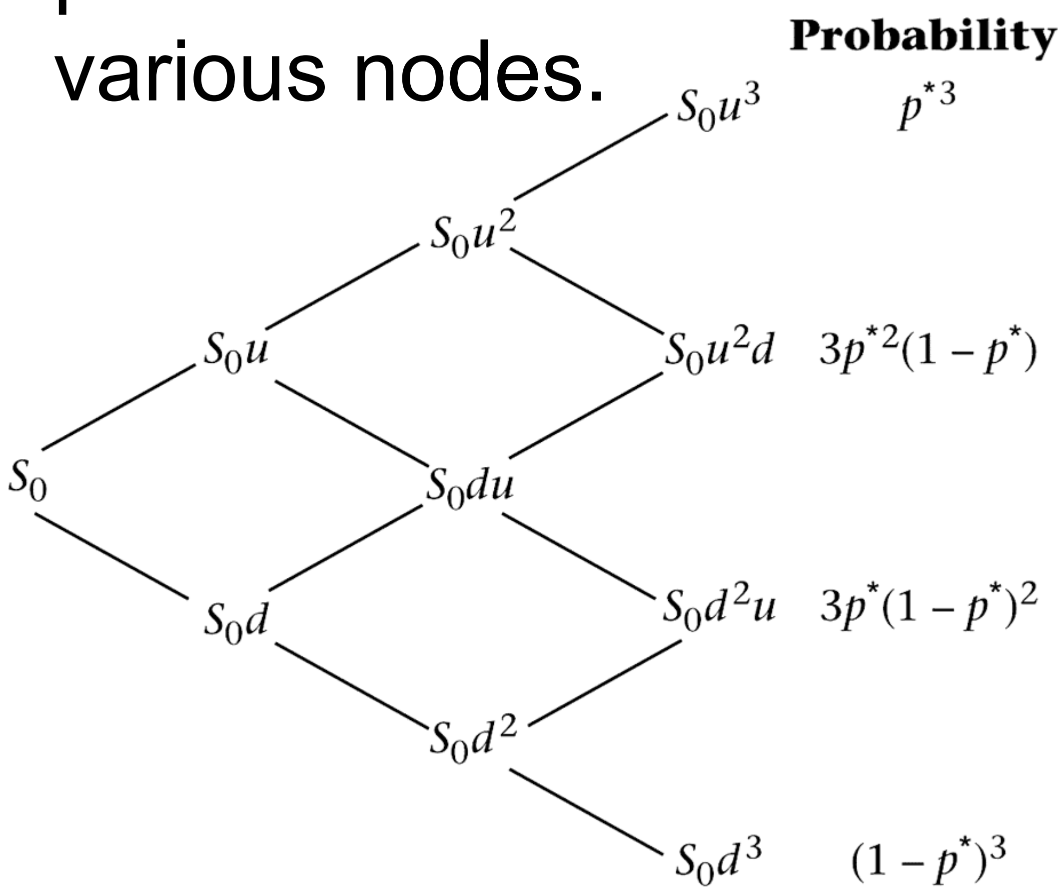
The Binomial Model (cont'd)

- Equation (11.6) solves the three problems in the random walk
  - The stock price cannot become negative
  - As *h* gets smaller, up and down moves get smaller
  - There is a $(r - \delta)h$ term, and we can choose the probability of an up move, so we can guarantee that the expected change in the stock price is positive

Lognormality and the Binomial Model

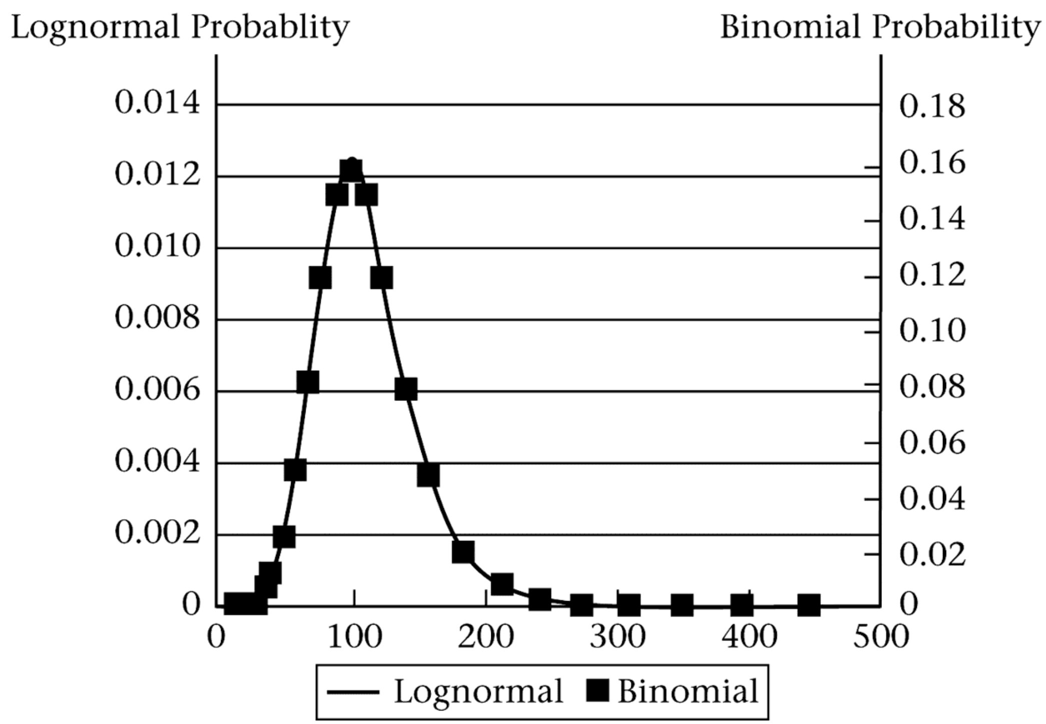
- The binomial tree approximates a lognormal distribution, which is commonly used to model stock prices
- The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed
- With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance of extremely high stock prices

- The binomial model implicitly assigns probabilities to the various nodes.



Lognormality and the Binomial Model (cont'd)

- The following graph compares the probability distribution for a 25-period binomial tree with the corresponding lognormal distribution



Alternative Binomial Trees

- There are other ways besides equation (11.6) to construct a binomial tree that approximates a lognormal distribution
  - An acceptable tree must match the standard deviation of the continuously compounded return on the asset and must generate an appropriate distribution as $h \to 0$
  - Different methods of constructing the binomial tree will result in different *u* and *d* stock movements
  - No matter how we construct the tree, to determine the risk-neutral probability, we use

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

and to determine the option value, we use

$$C = e^{-rh}[p^* C_u + (1 - p^*)C_d]$$

Alternative Binomial Trees (cont'd)

- **The Cox-Ross-Rubinstein binomial tree**
  - The tree is constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}} \qquad (11.18)$$

  - A problem with this approach is that if *h* is large or σ is small, it is possible that $e^{rh} > e^{\sigma\sqrt{h}}$. In this case, the binomial tree violates the restriction of

$$u > e^{(r-\delta)h} > d$$

  - In practice, *h* is usually small, so the above problem does not occur

Alternative Binomial Trees (cont'd)

- **The lognormal tree**
  - The tree is constructed as

$$u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} \qquad (11.19)$$

- Although the three different binomial models give different option prices for finite *n*, as $n \rightarrow \infty$ all three binomial trees approach the same price.

Is the Binomial Model Realistic?

- The binomial model is a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model here assumes that
  - Volatility is constant
  - “Large” stock price movements do not occur
  - Returns are independent over time
- All of these assumptions appear to be violated in the data