The Binomial Tree and Lognormality
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• The usefulness of the binomial pricing model hinges on the binomial tree providing a reasonable representation of the stock price distribution

• The binomial tree approximates a lognormal distribution
The Random Walk Model

• It is often said that stock prices follow a random walk

• Imagine that we flip a coin repeatedly
  – Let the random variable $Y$ denote the outcome of the flip
  – If the coin lands displaying a head, $Y = 1$; otherwise, $Y = -1$
  – If the probability of a head is $1/2$, we say the coin is fair
  – After $n$ flips, with the $i^{th}$ flip denoted $Y_i$, the cumulative total, $Z_n$, is

$$Z_n = \sum_{i=1}^{n} Y_i$$  (11.8)
The Random Walk Model (cont’d)

• We can represent the process followed by $Z_n$ in terms of the change in $Z_n$

\[ Z_n - Z_{n-1} = Y_n \]

or

Heads: $Z_n - Z_{n-1} = +1$

Tails: $Z_n - Z_{n-1} = -1$
The Random Walk Model (cont’d)

• A random walk, where with heads, the change in $Z$ is 1, and with tails, the change in $Z$ is $-1$. 

![Cumulative Counter, Z diagram](image)
The Random Walk Model (cont’d)

• The idea that asset prices should follow a random walk was articulated in Samuelson (1965)

• In efficient markets, an asset price should reflect all available information. In response to new information the price is equally likely to move up or down, as with the coin flip

• The price after a period of time is the initial price plus the cumulative up and down movements due to new information
Modeling Stock Prices
As a Random Walk

• The above description of a random walk is not a satisfactory description of stock price movements. There are at least three problems with this model
  – If by chance we get enough cumulative down movements, the stock price will become negative
  – The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price
  – The stock, on average, should have a positive return. However, the random walk model taken literally does not permit this

• The binomial model is a variant of the random walk model that solves all of these problems
Continuously Compounded Returns

• The binomial model assumes that continuously compounded returns are a random walk

• Some important properties of continuously compounded returns
  – The logarithmic function computes returns from prices
  – The exponential function computes prices from returns
  – Continuously compounded returns are additive
  – Continuously compounded returns can be less than −100%
The Standard Deviation of Returns

• Suppose the continuously compounded return over month $i$ is $r_{\text{monthly},i}$. The annual return is

$$r_{\text{annual}} = \sum_{i=1}^{12} r_{\text{monthly},i}$$

• The variance of the annual return is (11.14)

$$Var(r_{\text{annual}}) = Var\left(\sum_{i=1}^{12} r_{\text{monthly},i}\right)$$
The Standard Deviation of Returns (cont’d)

• Suppose that returns are uncorrelated over time and that each month has the same variance of returns. Then from equation (11.14) we have

\[ \sigma^2 = 12 \times \sigma^2_{\text{monthly}} \]

where \( \sigma^2 \) is the annual variance

• The annual standard deviation is

\[ \sigma_{\text{monthly}} = \frac{\sigma}{\sqrt{12}} \]

• If we split the year into \( n \) periods of length \( h \) (so that \( h = 1/n \)), the standard deviation over the period of length \( h \) is

\[ \sigma_h = \sigma \sqrt{h} \] (11.15)
The Binomial Model

• The binomial model is

\[ S_{t+h} = S_t e^{(r-\delta)h \pm \sigma \sqrt{h}} \]

• Taking logs, we obtain

\[ \ln(S_{t+h} / S_t) = (r - \delta)h \pm \sigma \sqrt{h} \]  \hspace{1cm} (11.16)

  – Since \( \ln(S_{t+h}/S_t) \) is the continuously compounded return from \( t \) to \( t+h \), the binomial model is simply a particular way to model the continuously compounded return
  
  – That return has two parts, one of which is certain, \( (r-\delta)h \), and the other of which is uncertain, \( \sigma \sqrt{h} \)
The Binomial Model (cont’d)

• Equation (11.6) solves the three problems in the random walk
  – The stock price cannot become negative
  – As $h$ gets smaller, up and down moves get smaller
  – There is a $(r - \delta)h$ term, and we can choose the probability of an up move, so we can guarantee that the expected change in the stock price is positive
Lognormality and the Binomial Model

- The binomial tree approximates a lognormal distribution, which is commonly used to model stock prices.

- The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed.

- With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance of extremely high stock prices.
• The binomial model implicitly assigns probabilities to the various nodes.

\[ \begin{aligned}
S_0 & \quad S_0u & \quad S_0u^2 & \quad S_0u^3 \\
S_0u & \quad S_0u^2d & \quad 3p^*_2(1-p^*) \\
S_0d & \quad S_0d^2u & \quad 3p^*(1-p^*)^2 \\
S_0d^2 & \quad S_0d^3 & \quad (1-p^*)^3 \\
\end{aligned} \]
The following graph compares the probability distribution for a 25-period binomial tree with the corresponding lognormal distribution.
Alternative Binomial Trees

• There are other ways besides equation (11.6) to construct a binomial tree that approximates a lognormal distribution
  
  − An acceptable tree must match the standard deviation of the continuously compounded return on the asset and must generate an appropriate distribution as $h \to 0$
  
  − Different methods of constructing the binomial tree will result in different $u$ and $d$ stock movements
  
  − No matter how we construct the tree, to determine the risk-neutral probability, we use
    \[
    p^* = \frac{e^{(r-\delta)h} - d}{u - d}
    \]
    and to determine the option value, we use
    \[
    C = e^{-rh} [ p^* C_u + (1 - p^*) C_d ]
    \]
Alternative Binomial Trees (cont’d)

• The Cox-Ross-Rubinstein binomial tree
  – The tree is constructed as
    \[
    u = e^{\sigma \sqrt{h}} \\
    d = e^{-\sigma \sqrt{h}} 
    \]
    \[(11.18)\]
  – A problem with this approach is that if \( h \) is large or \( \sigma \) is small, it is possible that \( e^{rh} > e^{\sigma \sqrt{h}} \). In this case, the binomial tree violates the restriction of
    \[
    u > e^{(r-\delta)h} > d 
    \]
  – In practice, \( h \) is usually small, so the above problem does not occur
Alternative Binomial Trees (cont’d)

• The lognormal tree
  – The tree is constructed as
    \[ u = e^{(r-\delta-0.5\sigma^2)h + \sigma\sqrt{h}} \]
    \[ d = e^{(r-\delta-0.5\sigma^2)h - \sigma\sqrt{h}} \] (11.19)

• Although the three different binomial models give different option prices for finite \( n \), as \( n \to \infty \) all three binomial trees approach the same price.
Is the Binomial Model Realistic?

• The binomial model is a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model here assumes that
  – Volatility is constant
  – “Large” stock price movements do not occur
  – Returns are independent over time

• All of these assumptions appear to be violated in the data