Probability Theory on Coin Toss Space

1. Finite Probability Spaces
2. Random Variables, Distributions, and Expectations
3. Conditional Expectations

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Inspiration

- A finite probability space is used to model the phenomena in which there are only finitely many possible outcomes
- Let us discuss the binomial model we have studied so far through a very simple example
- Suppose that we toss a coin 3 times; the set of all possible outcomes can be written as

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

- Assume that the *probability* of a head is p and the probability of a tail is $q = 1 - p$
- Assuming that the tosses are independent the probabilities of the elements $\omega = \omega_1\omega_2\omega_3$ of Ω are

$$\mathbb{P}[HHH] = p^3, \mathbb{P}[HHT] = \mathbb{P}[HTH] = \mathbb{P}[THH] = p^2 q,$$
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An Example (cont'd)

- The subsets of Ω are called events, e.g.,

$$\text{"The first toss is a head"} = \{\omega \in \Omega : \omega_1 = H\} = \{HHH, HTH, HTT\}$$

- The probability of an event is then

$$\mathbb{P}[\text{"The first toss is a head"}] = \mathbb{P}[HHH] + \mathbb{P}[HTH] + \mathbb{P}[HTT] = p$$

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Definitions

- A finite probability space consists of a sample space Ω and a probability measure $\mathbb{P}$.
  The sample space Ω is a *nonempty finite* set
  and the probability measure $\mathbb{P}$ is a *function* which assigns to each element ω in Ω a number in $[0,1]$ so that

$$\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

  An event is a *subset* of Ω.
  We define the probability of an event A as

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$$

- *Note:*

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  and if $A \cap B = \emptyset$

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Random variables

- **Definition.** A random variable is a real-valued function defined on Ω
- *Example (Stock prices)* Let the sample space Ω be the one corresponding to the three coin tosses. We define the stock prices on days $0, 1, 2$ as follows:

$$S_0(\omega_1\omega_2\omega_3) = 4 \text{ for all } \omega_1\omega_2\omega_3 \in \Omega$$

$$S_1(\omega_1\omega_2\omega_3) = \begin{cases} 8 & \text{for } \omega_1 = H \\ 2 & \text{for } \omega_1 = T \end{cases}$$

$$S_2(\omega_1\omega_2\omega_3) = \begin{cases} 16 & \text{for } \omega_1 = \omega_2 = H \\ 4 & \text{for } \omega_1 \neq \omega_2 \\ 1 & \text{for } \omega_1 = \omega_2 = H \end{cases}$$

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Distributions

- The distribution of a random variable is a specification of the probabilities that the random variable takes various values.
- Following up on the previous example, we have

$$\begin{aligned}\mathbb{P}[S_2 = 16] &= \mathbb{P}\{\omega \in \Omega : S_2(\omega) = 16\}\\ &= \mathbb{P}\{\omega = \omega_1\omega_2\omega_3 \in \Omega : \omega_1 = \omega_2\}\\ &= \mathbb{P}[HHH] + \mathbb{P}[HHT] = p^2\end{aligned}$$

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Expectations

- Let a random variable X be defined on a finite probability space $(\Omega, \mathbb{P})$. The expectation (or expected value) of X is defined as

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}[\omega]$$

- The variance of X is

$$Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

- *Note:* The expectation is *linear*, i.e., if X and Y are random variables on the same probability space and c and d are constants, then

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Back to the binomial pricing model

- The risk neutral probabilities were chosen as

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}, \; q^* = 1 - p^*$$

- Thus, at any time n and for any sequences of coin tosses (i.e., *paths* of the stock price) $\omega = \omega_1\omega_2\dots\omega_n$, we have that

$$S_n(\omega) = e^{-rh}\left[p^* S_{n+1}(\omega_1\dots\omega_n H) + q^* S_{n+1}(\omega_1\dots\omega_n T)\right]$$

- In words, the stock price at time n is the discounted weighted average of the two possible stock prices at time $n+1$, where p^* and q^* are the weights used in averaging
- Define

$$\mathbb{E}_n^*[S_{n+1}](\omega_1\dots\omega_n) = p^* S_{n+1}(\omega_1\dots\omega_n H) + q^* S_{n+1}(\omega_1\dots\omega_n T)$$

- Then, we can write

$$S_n = e^{-rh}\,\mathbb{E}_n^*[S_{n+1}]$$

- We call $\mathbb{E}_n^*[S_{n+1}]$ the conditional expectation of S_{n+1} based on the information known at time n

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The Definition

- Let $1 \leq n \leq N$ and let $\omega_1, \dots \omega_n$ be given and temporarily fixed. Denote by $\chi(\omega_{n+1} \dots \omega_N)$ the number of *heads* in the continuation $\omega_{n+1} \dots \omega_N$ and by $\tau(\omega_{n+1} \dots \omega_N)$ the number of *tails* in the continuation $\omega_{n+1} \dots \omega_N$

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Properties

$$\mathbb{E}_0^*[X] = \mathbb{E}^*[X],\ \mathbb{E}_N^*[X] = X$$

- *Linearity:*

$$\mathbb{E}_n[cX + dY] = c\mathbb{E}_n[X] + d\mathbb{E}_n[Y]$$

- *Taking out what is known:* If X actually depends on the first n coin tosses only, then

$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y]$$

- *Iterated conditioning:* If $0 \leq n \leq m \leq N$, then

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An illustration of the independence property

- In the same example that we have looked at so far, assume that the actual probability that the stock price rises in any given period equals $p = 2/3$ and consider

$$\mathbb{E}_1\left[S_2/S_1\right](H) = \frac{2}{3}\cdot\frac{S_2(HH)}{S_1(H)} + \frac{1}{3}\cdot\frac{S_2(HT)}{S_1(H)} = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}_1\left[S_2/S_1\right](T) = \frac{2}{3}\cdot\frac{S_2(TH)}{S_1(T)} + \frac{1}{3}\cdot\frac{S_2(TT)}{S_1(T)} = \cdots = \frac{3}{2}$$

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