M329F Theory of Interest
Spring 2020
University of Texas at Austin
In-Term Exam III
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Problem 3.1. *Source: Problem #37 from Course 2 Exam, May 2001.*
Roger borrows L for four years at an annual effective interest rate of 8%. The loan is supposed to be paid back using the amortization method with payments at the end of every year.

You are given that the outstanding loan balance at the end of the second year equals 1076.82 and that the outstanding loan balance at the end of the third year equals 559.12.

Based on these two values, find the amount of principal in the first payment.

(a) 443.84
(b) 838.78
(c) 1293.22
(d) 2000
(e) None of the above.

Solution: (a)
Let the amount of every level payment be denoted by P. Then,

$$L = Pa_{\overline{4}|}0.08 = P \cdot \frac{1-v^4}{i} = P \cdot 3.31213.$$

So, $P = \frac{L}{3.31213} = 0.301921L$.

There are multiple ways to write down the outstanding loan balance at the end of every payment period. Here is one

$$\begin{aligned}
k = 1 &: OLB_1 = L(1+i) - P = L(1.08 - 0.301921) = 0.778079L,\\
k = 2 &: OLB_2 = L(1+i)^2 - P(1+i) - P = L(1.08^2 - (1.08+1)0.301921) = 0.538404L,\\
k = 3 &: OLB_3 = Pv = 0.279556L,\\
K = 4 &: OLB_4 = 0.
\end{aligned}$$

From

$$OLB_3 = 0.279556L = 559.12,$$

we get $L \approx 2000$.

Or, from

$$OLB_2 = 0.538404L = 1076.82,$$

again, we get $L \approx 2000$.

Let the amount of principal in the first payment be denoted by P_1. Then,

$$P_1 = P - L \cdot i = 0.301921L - 0.08L = 0.221921L.$$

So,

$$P_1 = 0.221921 \cdot 2000 = 443.842.$$

Problem 3.2. Roger invests 1000 at the end of each year for seven years at an annual effective interest rate of 0.05.

The interest is credited at the end of every year, and Roger immediately seizes the opportunity to reinvest this interest into another account which accumulates according to an annual effective interest rate of 6%.

Find the total balance of both accounts at the end of seven years.

(a) 7732.51
(b) 8161.50
(c) 8462.34
(d) 9823.12
(e) None of the above.

Solution: (b)
The balance on the first account at the end of seven year is 7000.

The balance on the second account at the end of seven year is

$$Pi(1+j)^6 + 2Pi(1+j)^5 + \cdots + 6Pi = Pi[(1+j)^6 + 2(1+j)^5 + \cdots + 6] = Pi(Is)_{\overline{6|}\,j}$$

with $P = 1000, j = 0.06$ and $i = 0.05$. Once we plug in the given numbers, we get that the balance is

$$50 \cdot (Is)_{\overline{6|}\,0.06} = 50 \cdot \frac{\ddot{s}_{\overline{6|}\,0.06} - 6}{0.06} = 50 \cdot 23.23 = 1161.5.$$

So, the combined balance is

$$7000 + 1161.5 = 8161.5.$$

Problem 3.3. At the beginning of a quarter, Roger's aunt Agatha buys shares of preferred stock S at a price to provide her a yield of 8% convertible quarterly assuming that all the dividends are paid.

The stock S is nonadjustable and nonparticipating which implies that the dividend schedule which was in force at the time of purchase cannot be altered. So, the dividends are fixed at the times and levels that are stipulated at issuance, namely \$0.50 per share each quarter.

According to the dividend-discount model, what is the price that aunt Agatha should pay per share of stock?

(a) 6.25
(b) 12.50
(c) 17.75
(d) 25.00
(e) None of the above.

Solution: (d)
By the dividend discount model, the theoretical price of a share of stock is

$$P = 0.50 \cdot \frac{1}{j}$$

where j is the effective interest rate per quarter, i.e., $j = 0.08/4 = 0.02$. So, $P = 25$.

Problem 3.4. Assume that the price of a two-year zero-coupon bond is 0.881659 and that the price of a three-year zero-coupon bond is 0.816298. The two bonds have the same redemption amounts of $1.

Find $f_{[2,3]}$, i.e., the implied forward interest rate from year 2 to year 3.

(a) 0.06
(b) 0.07
(c) 0.08
(d) 0.09
(e) None of the above.

Solution: (c)
Let $P(0,2)$ denote the price of a two-year zero-coupon bond and let $P(0,3)$ denote the price of a two-year zero-coupon bond. Then,

$$f_{[2,3]} = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.881659}{0.816298} - 1 = 0.08.$$

Problem 3.5. Roger borrows 1000 for 10 years at an annual effective interest rate of 10%.

Roger chooses to repay the loan using a **sinking fund** that pays an annual effective rate of 14%.

He will be paying the amount $P = 162.75$ at the end of every year.

Determine the balance in the sinking fund at the end of the 10 years after the loan is repaid.

(a) About 113
(b) About 163
(c) About 213
(d) About 263
(e) None of the above

Solution: (c)
The annual interest payment is $1000 \cdot 0.1 = 100$. So, the annual deposit into the sinking funs is 62.745.

Hence, the accumulated value in the sinking fund account at the end of the ten years equals

$$62.745 \cdot s_{\overline{10|}\,0.14} = 1213.319.$$

Once the principal is repaid, the balance is about 213.319.

Problem 3.6. A 7%-bond with semiannual coupons is sold at a price of 79.30, while a 9%-bond with semiannual coupons is sold at a price of 93.10.

Both bonds have the face amount equal to 100 are redeemable in N years and have the same yield rate j. Find N.

(a) 10
(b) 12
(c) 14
(d) 16
(e) None of the above.

Solution: (b)
Using the premium-discount pricing formula, we get

$$79.30 = 100 + 100(0.035 - j)a_{\overline{n}|\,j},$$
$$93.10 = 100 + 100(0.045 - j)a_{\overline{n}|\,j}.$$

So,

$$\frac{79.30 - 100}{93.10 - 100} = \frac{0.035 - j}{0.045 - j}.$$

Hence, $j = 0.05$.

By the basic pricing formula, with n denoting the number of half-years in the term of the bond,

$$79.30 = 100 \cdot (1 + 0.05)^{-n} + 3.5 \cdot \frac{1 - (1 + 0.05)^{-n}}{0.05}.$$

Solving for n, we get $n = 24$. So, the number of years is $N = 12$.

Problem 3.7. Roger wants to be able to buy a perpetuity-immediate in exactly 10 years. This perpetuity should make payments equal to $1,000 at the end of every quarter.

To finance this purchase, Roger opens an account today and immediately starts depositing money continuously into that account. The rate at which the money is deposited is equal to R per annum.

Assume that the force of interest is constant and equal to $\delta = 9.5\%$.

Find the rate R such that at the end of 10 years, Roger has the exact amount of money needed to purchase the perpetuity-immediate.

(a) About 1,672
(b) About 2,045
(c) About 2,490
(d) About 3,012
(e) None of the above.

Solution: (c)
The fair price of the perpetuity is exactly

$$P = 1000 \cdot a_{\overline{\infty}|}\,j = \frac{1000}{j}$$

where j stands for the effective interest rate per quarter equivalent to the given force of interest δ. We have that

$$j = e^{\delta/4} - 1 \approx 2.4\%.$$

So,

$$P = 1000 \cdot a_{\overline{\infty}|}\,j \approx 41,607.24.$$

This price must equal the accumulated value at time 10 in the account Roger opens today. So,

$$P = X \cdot \bar{s}_{\overline{10}|} = R \cdot \frac{e^{10\delta} - 1}{\delta} \approx 16.72R.$$

So, $R \approx 2,490$.

Problem 3.8. Roger purchased a 20−year par value bond with semiannual coupons at a nominal annual rate of 8% semiannually. The price was 1000.

The bond can be **called** at par value X on any coupon date starting at the end of year 10.

Find this par value X if the following condition is given:

The purchase price of the bond guarantees that Roger will receive a nominal annual yield of at least 6% convertible semiannually.

(a) 870.4926

(b) 912.5291

(c) 985.7293

(d) 1000

(e) None of the above.

Solution: (a)
First, we need to determine the exercise (call) date of this bond that is the least profitable for Roger. It suffices to find out if the bond was sold at a premium or at a discount.

Note that the coupon rate is higher than the yield rate. This means that the bond was sold at a premium. Hence, the call date that would cause the lowest yield for Roger is the **earliest** possible date when the bond can be called. The problem statement tells us that this date is $n = 2 \cdot 10 = 20$ half-years (coupon periods) after the purchase of the bond.

Using the above number of coupon periods, i.e., setting $n = 20$ in the bond pricing formula, we get

$$1000 = X \cdot 0.04 \cdot a_{\overline{20}|\,0.03} + X \cdot \left(\frac{1}{1.03}\right)^{20}.$$

Thus,

$$X = \frac{1000}{0.04 \cdot a_{\overline{20}|\,0.03} + \left(\frac{1}{1.03}\right)^{20}} = \frac{1000}{0.04 \cdot 14.8775 + \left(\frac{1}{1.03}\right)^{20}} \approx 870.4926.$$

Problem 3.9. You are given the following table of spot rates:

Length of Investment	Spot rate
1 year	0.0400
2 years	0.0500
3 years	0.0575
4 years	0.0625
5 years	0.0650

Find the level swap interest rate for the interest-rate swap associated with a five-year loan for $100,000 with interest-only, annual, end-of-year payments.

(a) 0.0517

(b) 0.0589

(c) 0.0601

(d) 0.0639

(e) None of the above.

Solution: (d)
Let the prices of zero-coupon bonds redeemable at time$-k$ be denoted by $P(0,k)$ for $k = 1, \ldots, 5$. Then,

$$\begin{aligned} P(0,1) &= (1.04)^{-1} = 0.9615, \\ P(0,2) &= (1.05)^{-2} = 0.907, \\ P(0,3) &= (1.0575)^{-3} = 0.8456, \\ P(0,4) &= (1.0625)^{-4} = 0.7847, \\ P(0,5) &= (1.065)^{-5} = 0.7299. \end{aligned}$$

The sum of these prices is about 4.2287. So, the level swap rate R is

$$R = \frac{1 - P(0,5)}{\sum_{k=1}^{5} P(0,k)} = \frac{1 - 0.7299}{4.2287} = 0.0639.$$

Problem 3.10. An n-year \$2,000 par-value bond has 9% annual coupons. The book value of this bond at the end of the third year is \$1,850.
The book value of this bond at the end of the fifth year is \$1,900.
Let P denote the price of this bond. Then,

(a) $P \approx 1,665$

(b) $P \approx 1,793$

(c) $P \approx 2,013$

(d) $P \approx 2,203$

(e) None of the above

Solution: (b)
In our usual notation, from the definition of the book value (see equation (6.5.4) in the textbook) and using the given values of B_3 and B_5, we get

$$\begin{aligned} 1,900 = B_5 &= B_3(1+j)^2 - (Fr)(1+j) - (Fr) \\ &= 1,850(1+j)^2 - (2,000 \cdot 0.09)(1+j) - (2,000 \cdot 0.09) \\ &= 1,850(1+j)^2 - 180(1+j) - 180. \end{aligned}$$

Set $x = 1 + j$. Then x must satisfy

$$185x^2 - 18x - 208 = 0.$$

The only acceptable solution is $x \approx 1.11$ which yields $j = 0.11$. Thus, the price of the bond is

$$\begin{aligned} P &= 180 a_{\overline{3}|j} + B_3 (1+j)^{-3} \\ &= 180 \frac{1 - 1.11^{-3}}{0.11} + 1,850 (1.11)^{-3} \\ &\approx 1,792.5727. \end{aligned}$$

Problem 3.11. *Source: Course 2, November 2000, Problem #30.*
A 1000-par-value 20-year bond with annual coupons and redeemable at maturity for 1050 is purchased for P to yield an annual effective rate of 8.25% . The amount of the first coupon is 75. Each subsequent coupon is 3% greater than the preceding coupon. Determine the purchase price of the bond P.

(a) About 985
(b) About 1000
(c) About 1050
(d) About 1115
(e) None of the above

Solution: (d)
Using the formulae we developed for the annuities with geometrically increasing payments, we get that the price is

$$P = C_1 \cdot \frac{1 - (\frac{1+g}{1+j})^n}{j - g} + C(1 + j)^{-n}$$

with the first coupon amount equal to $C_1 = 75$, the percentage increase in coupon amounts equal to $g = 0.03$, the yield rate $j = 0.0825$, the total number of coupons $n = 20$, and the redemption amount $C = 1050$.
We obtain $P \approx 1115.11$.

Problem 3.12. *Source: FM exam, May 2003, Problem #42.*
Consider a 10,000 par value bond redeemable at par in 10 years. Its coupons are annual at a rate of 8%. This bonds is purchased today at a premium to yield an annual effective interest rate of 6%. Find the interest portion of the 7^{th} coupon.

(a) 632.12
(b) 641.58
(c) 651.24
(d) 660.01
(e) None of the above.

Solution: (b)
In our usual notation,

$$I_7 = 0.06B_6.$$

Each coupon is equal to $10000 \cdot 0.08 = 800$. Using the prospective method, we can get the book value of the bond right after the 6^{th} coupon is paid:

$$B_6 = (800)a_{\overline{4}|0.06} + 10000(1.06)^{-4} = 10693.02112.$$

Finally, $I_7 = 641.5812674$.