M329F *Theory of Interest*
Spring 2020
University of Texas at Austin
In-Term Exam II
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Notes: This is a closed book and closed notes exam. The maximal score on this exam is 50 points.
Time: 50 minutes

Problem 1.1. (5 points) You make an initial investment of \$1,000 and an additional investment of \$500 at time 1.
The balance at time 1, just before the deposit is made equals \$1,200, while the final balance at time 2 equals \$2,000.

Find the (approximate) dollar-weighted rate of return **per annum**.

(a) 0.09
(b) 0.12
(c) 0.15
(d) 0.18
(e) 0.21

Solution: (d)
In the notation of section 2.6 in the textbook,

$$A = 1000, B = 2000, \quad C = 500.$$

$$I = B - A - C = 2000 - 1000 - 500 = 500.$$

The two-year (approximate) dollar-weighted rate of return is

$$j = \frac{500}{1000 + 0.5 \cdot 500} = \frac{500}{1250} = 0.4.$$

So, the one-year (approximate) dollar-weighted rate of return is

$$i = \sqrt{(1 + 0.4)} - 1 \approx 0.1832.$$

Problem 1.2. (5 points) Assume compound interest and let a basic perpetuity-due have the present value equal to 20. Another perpetuity-due makes equal payments of R at the beginning of every two years. Find the value R such that the two perpetuities are exchangeable without gain or loss, i.e., such that the two perpetuities have the same present value.

(a) 1.95
(b) 2

(c) 2.05

(d) 2.10

(e) 2.15

Solution: (a)
Let d denote the effective annual rate of discount. Then, the given present value of the basic perpetuity-due yields

$$20 = \frac{1}{d} \Rightarrow d = 0.05.$$

On the other hand, we can express the present value of the other perpetuity as

$$20 = \frac{R}{\tilde{d}}$$

where $\tilde{d}$ denotes the effective rate of discount for two years.

Recall that d and $\tilde{d}$ are connected by the following equality

$$1 - d = (1 - \tilde{d})^{1/2} \Rightarrow \tilde{d} = 1 - (1 - d)^2 = (1 - (1 - d))(1 + 1 - d) = 0.05 \cdot 1.95 = 0.0975.$$

So,

$$R = 20\tilde{d} = 1.95.$$

Problem 1.3. (5 points) A 5-year annuity-immediate pays 100 the first year and each subsequent payment is 2% larger than the one preceding it. Find the present value of this annuity if the effective annual interest rate equals $i = 5\%$.

(a) 440.75

(b) 442.75

(c) 449.75

(d) 453.75

(e) 468.75

Solution: (c)
With $g = 0.02$ and $v = 1.05^{-1}$, the present value of this annuity can be expressed as

$$100v + 100(1+g)v^2 + \cdots + 100(1+g)^4 v^5 = 100 \cdot \frac{1 - (\frac{1.02}{1.05})^5}{0.05 - 0.02} \approx 449.746.$$

Problem 1.4. (5 points) *Source: SoA, November 1996, Problem #4.*
Alice and Bob shared **equally** in an inheritance. Using her inheritance, Alice immediately bought a ten-year annuity-due with annual payments of \$2,500 each. Alice's annuity yields 8% effective per annum. Bob put his inheritance in an investment fund earning an annual effective interest rate of 9% and left it there for two years. Then, he withdrew the balance and bought a 15−year annuity immediate with annual payments equal to X. Bob's annuity yields 8% effective per annum. How much is X?

(a) 2330.76
(b) 2474.76
(c) 2514.76
(d) 2565.76
(e) 2715.76

Solution: (c)
The present value of Alice's annuity due is

$$2500\ddot{a}_{\overline{10}|\,0.08} = 18117.21978.$$

Since they shared in the inheritance equally, this is also the amount of money that Bob gets. After two years in the account earning 9% effective annually, his balance is

$$(1.09)^2(18117.21978) = 21525.06882.$$

With this money he buys an annuity immediate with a 15−year term with each payment equal to X which yields him 8%. This means that

$$Xa_{\overline{15}|\,0.08} = 21525.06882 \quad \Rightarrow \quad X = \frac{21525.06882}{8.559479} = 2514.763995.$$

Problem 1.5. (5 pts) *Source: FM exam, May 2001, Problem #17.*
Let the **annual** effective interest rate be denoted by $i > 0$. Consider a perpetuity paying 10 at the end of each three-year period with the first payment at the end of year six. The present value of this perpetuity at time−0 equals 32 under the interest rate i. Calculate i.

(a) 0.044
(b) 0.055
(c) 0.066
(d) 0.077
(e) 0.088

Solution: (d)
With $v = \frac{1}{1+i}$, the present value of this perpetuity is

$$32 = 10(v^6 + v^9 + \dots) = \frac{10v^6}{1-v^3} \quad \Rightarrow \quad 5(v^3)^2 + 16v^3 - 16 = 0.$$

We solve the quadratic in v^3, keep the positive solution and get $v^3 = 0.8$. Hence,

$$(1+i)^3 = \frac{1}{0.8} = 1.25 \quad \Rightarrow \quad i = 0.077217$$

Problem 1.6. (5 points) An investor invests a certain amount of money at time 0 into an account governed by the time-varying force of interest $\delta_t = 0.035t$. How many years will it take the investor to quadruple her money (round your answer to the nearest integer)?

(a) About 3 years
(b) About 5 years
(c) About 7 years
(d) About 9 years
(e) None of the above

Solution: (d)
Assume that the principal is a single dollar, and denote the unknown length of time by n. Then, the equation of value is

$$e^{\int_0^n 0.035t\,dt} = 4 \Rightarrow 0.035n^2 = 2\ln(4).$$

So, $n \approx 8.9004$.

Problem 1.7. (5 points)*Source: SoA, May 1987, Problem #1.*
Let the annual effective interest rate be of 12.55% be charged on a loan of \$1000. The loan is to be repaid in three payments:

- 400 at the end of the first year,
- 800 at the end of the fifth year, and
- the balance X at the end of the tenth year.

What is X?

(a) 587.52
(b) 657.72
(c) 737.82
(d) 777.32
(e) 812.12

Solution: (b)
The equation of value at time−10 is

$$1000(1.1255)^{10} = 400(1.1255)^9 + 800(1.1255)^5 + X.$$

So,

$$X = 1000(1.1255)^{10} - 400(1.1255)^9 + 800(1.1255)^5 = 657.72$$

Problem 1.8. Bertie invests \$5,000 today and in return he gets:

- \$2,000 in one year,
- \$1,000 in two years, and
- \$3,000 in three years.

What is the annual effective yield rate on Bertie's investment?

(a) 0.0456

(b) 0.0677
(c) 0.0744
(d) 0.0893
(e) 0.1223

Solution: (d)
Here, we have to use our financial calculator. We set up the cashflow worksheet so that:

$$CFo = -5000, C01 = 2000, F01 = 1, C02 = 1000, F02 = 1, C03 = 3000, F03 = 1.$$

We CPT the IRR and get the yield rate of 8.9281595%

Problem 1.9. On January 1^{st}, an investment fund is worth 100. On May 1^{st}, its value is 120 and a withdrawal $W > 0$ is made. Then, on November 1^{st} of the same year, the value is again 100 and the same amount W is deposited. Finally, on January 1^{st} next year, the value of the fund is again equal to 100.

You are informed that the time-weighted rate of interest for the above set of transactions is 0%. Find the unknown deposit/withdrawal amount $W > 0$.

(a) 20
(b) 40
(c) 50
(d) 60
(e) 80

Solution: (a)
The formula for the time-weighted rate of return along with the conditions of the problem yields

$$\frac{120}{100} \cdot \frac{100}{120 - W} \cdot \frac{100}{100 + W} = 1 + 0.$$

Hence,

$$\frac{100}{120 - W} \cdot \frac{100}{100 + W} = \frac{100}{120}.$$

So,

$$\frac{100}{12,000 + 20W - W^2} = \frac{1}{120},$$

i.e.,

$$12,000 = 12,000 + 20W - W^2 \Rightarrow W(W - 20) = 0.$$

We have assumed that $W > 0$, so $W = 20$

Problem 1.10. At an annual effective interest rate i, the present value of a $50-$year annuity immediate which pays \$100 at the end of every year is equal to \$1379.31. What is the interest rate i?

(a) 0.04

(b) 0.05

(c) 0.06

(d) 0.07

(e) 0.08

Solution: (d)
Using the business calculator's CashFlow Worksheet with

$$CFo = -1379.31, \quad C01 = 100, \quad F01 = 50,$$

we hit IRR CPT and get about 7%.

Problem 1.11. (5 points) Consider an annuity immediate with the following regime of payments:

- end-of year payments equal to 10 for 5 years;
- end-of year payments equal to 15 for the following 5 years;
- end-of year payments equal to 10 for the following 3 years;
- end-of year payments equal to 6 for the following 2 years.

Let the present value of the above annuity immediate be denoted by P. In standard actuarial notation, you are given that

$$a_{\overline{5}|} = 4.4518, \quad a_{\overline{10}|} = 8.1109, \quad a_{\overline{13}|} = 9.9856, \quad \text{and} \quad a_{\overline{15}|} = 11.1184.$$

Find X.

(a) 94.32

(b) 98.96

(c) 102.56

(d) 112.34

(e) 124.95

Solution: (e)
We can rewrite the present value of our non-level annuity in terms of deferred annuities as follows:

$$\begin{aligned} X &= 10a_{\overline{5}|} + 15(a_{\overline{10}|} - a_{\overline{5}|}) + 10(a_{\overline{13}|} - a_{\overline{10}|}) + 6(a_{\overline{15}|} - a_{\overline{13}|}) \\ &= 6a_{\overline{15}|} + 4a_{\overline{13}|} + 5a_{\overline{10}|} - 5a_{\overline{5}|} \\ &= 6(11.1184) + 4(9.9856) + 5(8.1109) - 5(4.4518) = 124.9483 \end{aligned}$$

Problem 1.12. (5 points) Bertie borrows \$5,000 from Tuppy for a term of four years. Bertie agrees to pay interest at the end of each year at an annual effective interest rate of 4% and to repay the entire \$5,000 as a lump sum at the end of four years. Immediately after the third payment, Tuppy sells his right to future payments to Freddie at a price that will yield Freddie an effective annual rate of 3%. Let Tuppy's overall effective annual yield rate be denoted by y. How much is y?

(a) 0.01

(b) 0.015

(c) 0.02

(d) 0.3

(e) 0.35

Solution: CREDIT FOR ALL

From Tuppy's point of view, the cashflows are

- -5000 at time-0 (the original loan amount),
- $(0.04)5000 = 200$ at time-1, time-2, and time-3 (these are the interest-only payments from Bertie),
- the outstanding loan balance OLB_3 at the interest rate of 3% received at time-3 (from Freddie).

So, let's calculate OLB_3. We have

$$OLB_3 = 5200(1.03)^{-1} = 5048.543689.$$

Let us denote Tuppy's yield rate by y and set $v_y = \frac{1}{1+y}$. Then, y must satisfy

$$5000 = 200a_{\overline{3|}\,y} + 5048.543689v_y^3.$$

It's best to use the financial calculator at this point. I prefer to use the CashFlow Worksheet. In it, I set up

$$CFo = -5000, C01 = 200, F01 = 2, C02 = 5248.543689, F02 = 1.$$

Then, we CPT the IRR and get $y = 0.0431$.