*12. M329F: February 10th, 2020.

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest for the first 10 years is credited at a nominal discount rate of d compounded quarterly, and thereafter at a nominal interest rate of 6% compounded semiannually. The accumulated balance in the fund at the end of 30 years is 100.

Calculate d.

(A) 4.33%

(B) 4.43%

(C) 4.53%

(D) 4.63%

(E) 4.73%

13.

Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest

$\delta_t = \dfrac{t^2}{100}$, $t > 0$.

The amount of interest earned from time 3 to time 6 is also X.

Calculate X.

(A) 385

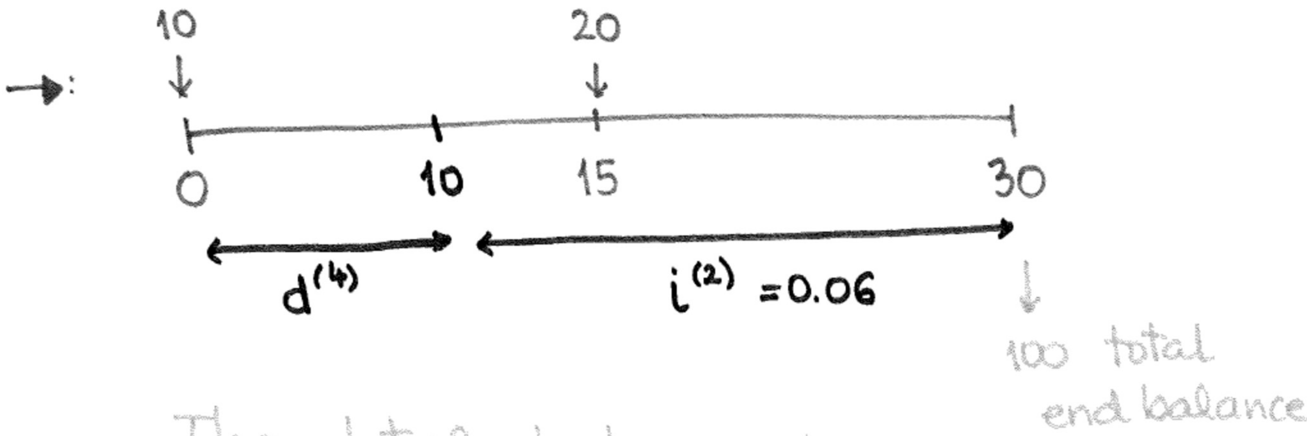
(B) 485

(C) 585

(D) 685

(E) 785

1.



The total balance is the result of the accumulation of:

- the 10 deposited @ time 0

and

- the 20 deposited @ time 15

The accumulated value of the 10 deposited @ time 0 is:

$$10\left(1-\frac{d^{(4)}}{4}\right)^{-10(4)}\left(1+\frac{i^{(2)}}{2}\right)^{20\cdot(2)}$$

The accumulated value of the 20 deposited @ time 15 is:

$$20\left(1+\frac{i^{(2)}}{2}\right)^{15(2)} =$$

$$= 20(1.03)^{30}$$

$$10\left(1-\frac{d^{(4)}}{4}\right)^{-40}(1.03)^{40}+20(1.03)^{30}=100$$

$$10\left(1-\frac{d^{(4)}}{4}\right)^{-40}=\frac{100-20(1.03)^{30}}{(1.03)^{40}}\approx 15.77$$

$$1-\frac{d^{(4)}}{4}=(1.577)^{-\frac{1}{40}}\approx 0.989$$

$$\Rightarrow \quad d^{(4)}=(1-0.989)\cdot 4\approx 0.0453$$

94.

A couple decides to save money for their child's first year college tuition.

The parents will deposit 1700 n months from today and another 3400 $2n$ months from today.

All deposits earn interest at a nominal annual rate of 7.2%, compounded monthly.

Calculate the maximum integral value of n such that the parents will have accumulated at least 6500 five years from today.

(A) 11

(B) 12

(C) 18

(D) 24

(E) 25

* 95.

Let S be the accumulated value of 1000 invested for two years at a nominal annual rate of discount d convertible semiannually, which is equivalent to an annual effective interest rate of i. $d^{(2)}$

Let T be the accumulated value of 1000 invested for one year at a nominal annual rate of discount d convertible quarterly. $d^{(4)}$

$S / T = (39/38)^4$.

Calculate i.

(A) 10.0%

(B) 10.3%

(C) 10.8%

(D) 10.9%

(E) 11.1%

3.

→: First, we figure out d:

$$S = 1000\left(1-\frac{d}{2}\right)^{-2\cdot 2} = 1000\left(1-\frac{d}{2}\right)^{-4}$$

$$T = 1000\left(1-\frac{d}{4}\right)^{-1\cdot 4} = 1000\left(1-\frac{d}{4}\right)^{-4}$$

Given: $$\frac{S}{T} = \left(\frac{39}{38}\right)^{\cancel{4}} = \frac{1000\left(1-\frac{d}{2}\right)^{-\cancel{4}}}{1000\left(1-\frac{d}{4}\right)^{-\cancel{4}}}$$

$$\Rightarrow \quad \frac{39}{38} = \frac{1-\frac{d}{4}}{1-\frac{d}{2}}$$

$$\Rightarrow \quad 39 - 39\cdot\frac{d}{2} = 38 - 38\cdot\frac{d}{4}$$

$$\Rightarrow \quad d\left(\frac{39}{2} - \frac{38}{4}\right) = 1$$

$$\Rightarrow \quad d = 0.10$$

Solve for i in:

$$1+i = \left(1-\frac{0.10}{2}\right)^{-2} = (1-0.05)^{-2} = 0.95^{-2}$$

$$\Rightarrow i = 0.108 \Rightarrow (c) \blacksquare$$

* Bruce deposits 100 into a bank account. His account is credited interest at an annual nominal rate of interest of 4% convertible semiannually.

At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at an annual force of interest of δ.

After 7.25 years, the value of each account is the same.

Calculate δ.

$$(1+0.02)^{14.5} = e^{\delta \cdot 7.25}$$

$$\Rightarrow (1+0.02)^2 = e^{\delta}$$

$$\Rightarrow \delta = 2\ln(1.02) = 0.0396 \Rightarrow (C)$$

(A) 0.0388

(B) 0.0392

(C) 0.0396

(D) 0.0404

(E) 0.0414

2.

Kathryn deposits 100 into an account at the beginning of each 4-year period for 40 years. The account credits interest at an annual effective interest rate of i.

The accumulated amount in the account at the end of 40 years is X, which is 5 times the accumulated amount in the account at the end of 20 years.

Calculate X.

(A) 4695

(B) 5070

(C) 5445

(D) 5820

(E) 6195

5.

Review: FORCE OF INTEREST

b_0... the initial balance

$b(t)$... time-t balance

Assume: $b'(t) = \delta b(t)$

$\uparrow$ force of interest

$$\Rightarrow \boxed{b(t) = b_0 e^{\delta \cdot t}}$$

The TIME-VARYING FORCE OF INTEREST

b_0 ... the initial balance

$b(t)$ ... time-t balance

<u>Assume</u>: the rate of change is

$$b'(t) = \delta_t \cdot b(t)$$

$\uparrow$ the <u>time-varying</u> force of interest

$$\rightarrow: \quad \frac{db(t)}{dt} = \delta_t b(t)$$

$$\frac{db(t)}{b(t)} = \delta_t dt$$

$$\ln(b(t)) = \int_0^t \delta_u du + c$$

$$\Rightarrow \quad b(t) = b_0 \cdot e^{\int_0^t \delta_u du}$$

If the initial balance is 1,
then we get the accumulation f'tion

$$a(t) = e^{\int_0^t \delta_u \, du}$$

We can reverse the process:
At time t, the force of interest is:

$$\ln(a(t)) = \int_0^t \delta_u \, du$$

$$\delta_t = \frac{d}{dt}\left[\ln(a(t))\right] = \frac{a'(t)}{a(t)}$$

Example. [FINDING δ_t from $a(t)$]

I. $a(t) = (1+0.05)^t (1+0.02)^{t/2}$

Then,

$$\delta_t = \frac{d}{dt}\left[\ln(a(t))\right]$$
$$= \frac{d}{dt}\left[t \cdot \ln(1.05) + \frac{t}{2}\ln(1.02)\right]$$
$$= \ln(1.05) + \frac{1}{2}\ln(1.02)$$

II. $a(t) = (1+0.05)^{t^2}$

Then,

$$\delta_t = \frac{d}{dt}\left[\ln(a(t))\right]$$
$$= \frac{d}{dt}\left[t^2 \cdot \ln(1.05)\right]$$
$$= 2t\ln(1.05)$$ NOT a constant