M329F: February 5^{th}, 2020.

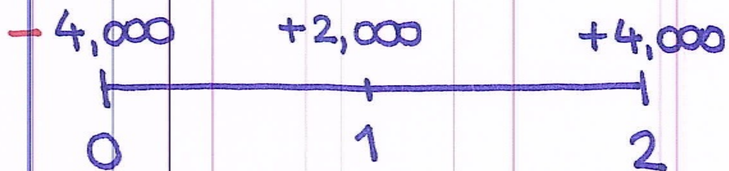
Problem. Roger initially deposits \$4,000 in an investment fund which pays him \$2,000 @ time·1 and \$4,000 @ time·2.

Sally gets \$2,000 @ time·0 and \$4,000 @ time·1; she deposits \$5,460 @ time·2 in return.

Both investments are governed by compound interest w/ the same annual effective interest rate i and they have the same net present values.

Calculate i.

→: Focus on Roger:



$$\Rightarrow NPV(R) = -4{,}000 + 2000\,v(1) + 4000 \cdot v(2)$$

$$= -4000 + 2000\,v + 4000\,v^2$$

$v(t) = v^t$ w/ $v = \frac{1}{1+i}$

Focus on Sally:

0	1	2
+2,000	+4000	−5460

$\Rightarrow$

$$NPV(S) = 2000 + 4000 \cdot v + (-5460) \cdot v^2$$

Given: $NPV(R) = NPV(S)$

$$-4000 + 2000v + 4000v^2 =$$
$$= 2000 + 4000v - 5460v^2$$

$$\Rightarrow 9460v^2 - 2000v - 6000 = 0$$

$$v_{1,2} = \frac{2000 \pm \sqrt{(2000)^2 + 4 \cdot 9460 \cdot 6000}}{2(9460)}$$

We only keep: $v = 0.9090\ldots$

$$\Rightarrow i = 0.1$$

Note: The NPV is:

$$-4000 + 2000(1.1)^{-1} + 4000(1.1)^{-2} =$$
$$= 1{,}123.9669$$

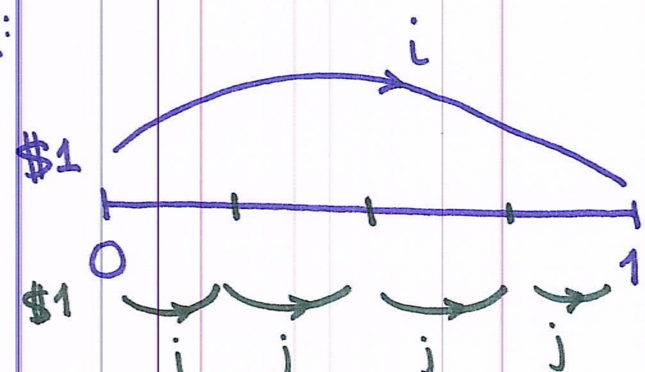
$i^{(m)}$ ... nominal interest rate convertible m^{th}-ly

$\Rightarrow$ $j := \dfrac{i^{(m)}}{m}$ is the <u>effective</u> interest rate for $\frac{1}{m}$ of the year

e.g., $i^{(2)}$ ... nominal interest rate convertible semiannually

$\Rightarrow$ $\dfrac{i^{(2)}}{2}$ ... effective interest rate per half-year

<u>Note</u>:



$1+i = (1+j)^m = \left(1+\dfrac{i^{(m)}}{m}\right)^m$

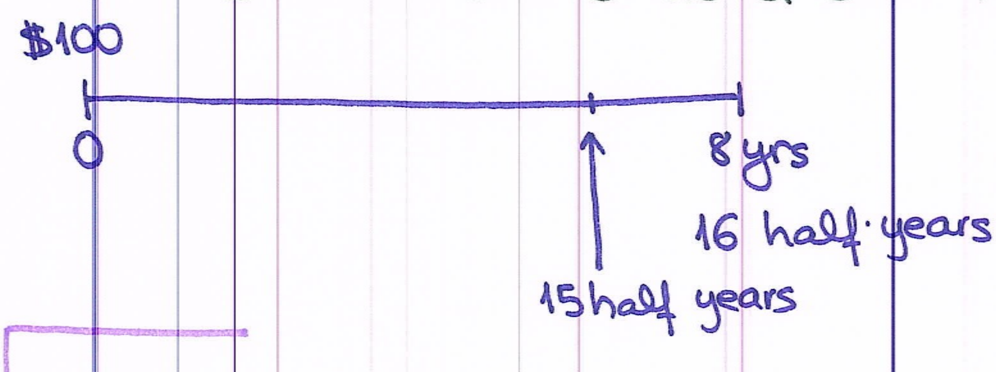
w/ $j = \dfrac{i^{(m)}}{m}$

<u>Now</u>:

$$1+i = (1-d)^{-1} = \left(1+\frac{i^{(m)}}{m}\right)^m$$

Problem. Roger deposits \$100 into an account which pays interest @ a nominal rate of x compounded semiannually.
At the same time, Sally deposits \$200 into a different account. Her account is governed by simple interest w/ an annual rate x.
No subsequent actions are taken. The AMOUNT of INTEREST they earned during the last six months of the 8^{th} year are equal.
Find x.

→: Look @ HALF-YEARS as a basic time unit.



effective interest rate over this time period

⇒ amount of interest during the time period
$=$ (balance in the beginning of time period) × (effective interest rate for that period)

In this problem, the effective interest rate per half·year is $\frac{x}{2}$.

$\Rightarrow$ Roger's balance @ the end of 15 half·yrs:

$$100\left(1+\frac{x}{2}\right)^{15}$$

$\Rightarrow$ His amt of interest in the 16^{th} half year:

$$\boxed{100\left(1+\frac{x}{2}\right)^{15}\cdot\frac{x}{2}}$$

In Sally's case, we have simple interest. So, no interest earned earns interest, i.e., only her principal earns interest.

$\Rightarrow$ During the 16^{th} half·year, she earns:

$$\boxed{200\cdot x\cdot\frac{1}{2}} \quad \text{in interest}$$

Finally: We equate the two amounts:

$$100\left(1+\frac{x}{2}\right)^{15}\left(\cancel{\frac{x}{2}}\right)=200\left(\cancel{\frac{x}{2}}\right) \quad /:100$$

$$\Rightarrow \left(1+\frac{x}{2}\right)^{15}=2 \quad \Rightarrow \boxed{x\approx 0.094588}$$

■

5.

<u>Note</u>: For $m = \frac{1}{n}$ $\quad n = 1, 2, \ldots$

$\frac{i^{(m)}}{m}$ ... effective interest rate for an n·year period,

e.g., $m = \frac{1}{5}$

$\frac{i^{(1/5)}}{1/5}$ ... effective interest rate per five year periods