Name:

M329F Theory of Interest
Fall 2020
University of Texas at Austin
Solution: In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 60 points.
Time: 50 minutes

TRUE/FALSE

1.3 (2)	TRUE	FALSE
1.4 (2)	TRUE	FALSE
1.5 (2)	TRUE	FALSE
1.6 (2)	TRUE	FALSE

MULTIPLE CHOICE

1.11 (3)	a	b	c	d	e
1.12 (3)	a	b	c	d	e
1.13 (3)	a	b	c	d	e

FOR GRADER'S USE ONLY:

DEF'NS	T/F	1.7	1.8	1.9	1.10	M.C.	Σ

1.1. DEFINITIONS.

Problem 1.1. (5 points) Provide the definition of an *effective interest rate for the time period* $[t_1, t_2]$ in terms of the accumulation function $a(\cdot)$.

Solution: The effective interest rate for the time period $[t_1, t_2]$ is

$$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}$$

Problem 1.2. (5 points) Provide the definition of the *force of interest* in terms of the accumulation function $a(\cdot)$ in the compound interest case.

Solution: The force of interest is given by (any of the following):

$$\delta = \frac{a'(t)}{a(t)} = \frac{d}{dt}[\ln(a(t))] = \ln(1+i).$$

1.2. **TRUE/FALSE QUESTIONS.** *Please, note your final answer on the front page.*

Problem 1.3. (2 pts) In our usual notation, for equivalent $d^{(n)}$ and $i^{(p)}$, we have that $d^{(n)} > i^{(p)}$. *True or false?*

Solution: FALSE
See Important fact 1.11.5 in the textbook.

Problem 1.4. (2 pts) Let the amount function A_K have the form

$$A_K(t) = \alpha(t+1)^3 + \beta(t+1).$$

Then, we know that $\alpha + \beta = K$. *True or false?*

Solution: TRUE

$$K = A_K(0) = \alpha + \beta.$$

Problem 1.5. (2 pts) If we wish to invest the amount X at a future time t_1 in order to have \$S at time $t_2 > t_1$, we should invest $X = S\,\frac{a(t_2)}{a(t_1)}$. *True or false?*

Solution: FALSE
See Important fact 1.7.4 in the textbook.

Problem 1.6. (2 pts) In our usual notation, $iv = d$. *True or false?*

Solution: TRUE

1.3. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.7. (4 points) *Source: Problem 1.3.1 from the textbook.*
You are given that, in our usual notation,

$$A_K(t) = \frac{1,000}{100-t} \quad 0 \leq t < 100.$$

(i) (2 pts) Find K.
(ii) (2 pts) Find $a(20)$.

Solution: We know that always $A_K(0) = K$. So, in the present problem,

$$K = A_K(0) = \tfrac{1000}{100} = 10.$$

The accumulation function equals

$$a(t) = \tfrac{1}{K}A_K(t) = \tfrac{1}{10}\cdot\tfrac{1000}{100-t} = \tfrac{100}{100-t} \text{ for every } t \in [0,100).$$

So,

$$a(20) = \frac{100}{100-20} = \frac{100}{80} = 1.25.$$

Problem 1.8. (8 points) *Source: Problem 1.8.5 from the textbook.*
Suppose you invest \$300 in a fund earning simple interest at the simple interest rate of 6%. Three years later, you withdraw the entire balance from that account and invest it in another fund earning 8% simple discount.

(i) (2 pts) What is the balance at the end of the first three years in the simple interest account?

(ii) (3 pts) How much time (including the three years in the simple interest account) will be required for the original \$300 to accumulate to \$650?

(iii) (3 pts) At what annual effective rate of compound interest would \$300 accumulate to \$650 in the same amount of time? In other words, what is the yield rate of this investment?

Solution:

(i) At the end of three years, the balance in the simple interest account is equal to

$$300(1 + 0.06 \cdot 3) = 354.$$

(ii) So, at time $T + 3$, the balance in the simple discount account equals

$$354(1 - 0.08T)^{-1}.$$

We need to solve for T in the equation

$$354(1 - 0.08T)^{-1} = 650.$$

We get $T \approx 5.69$. So, the time we are looking for is $T + 3 \approx 8.69$.

(iii) We need to solve for i in

$$300(1 + i)^{8.69} = 650.$$

We get $i \approx 0.093$.

Problem 1.9. (10 pts) Roger won in Bingo tournament. He has the following two options to collect his reward:

i. Payments \$100 now and \$108.15 in two years,

ii. A single payment of \$208 exactly one year from now.

Roger is good at interest theory and realizes that there are two effective annual interest rates i and j such that he would be indifferent between the above two scenarios under those two rates, i.e., the net present values of the two cash-flow streams would be equal if either i or j is used. Find $|i-j|$.

Solution: Equating the two present values, we get that the discount factor v should satisfy the following quadratic equation:

$$100 + 108.15v^2 = 208v.$$

We get to (sensible) solutions to the above equation: $v_1 = 206/216.3$ and $v_2 = 210/216.3$ yielding $i = 0.03$ and $j = 0.05$. So, $|i-j| = 0.02$.

Problem 1.10. (5 points) *Source: Problem 1.9.2 from the textbook.*
Roger wishes to obtain \$4,000 to pay her college tuition now. He qualifies for a loan with an annual effective discount rate of 3.5%.

(a) (3 pts) How much will he have to repay if the loan term is six years?

(b) (2 pts) What is the annual effective **interest** rate charged on this loan?

Solution:

(a) The accumulation function reads as

$$a(t) = (1 - 0.035)^{-t} \qquad \text{for } t \geq 0.$$

So, he will have to repay

$$4000a(6) = 4593.32.$$

(b)

$$i = \frac{d}{1-d} = \frac{0.035}{0.965} \approx 0.0363. \tag{1.1}$$

1.4. **MULTIPLE CHOICE QUESTIONS.** *Please, note your final answers on the front page.*

Problem 1.11. (5 pts) *Source: Exam FM/2, May 2005, Problem #7.*
Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The net present value of these cash flows is 364.46 at an annual effective interest rate i. Calculate i.

(a) About 10%
(b) About 11%
(c) About 12%
(d) About 13%
(e) None of the above

Solution: (a)
The condition on the NPV is

$$364.46 = 100 + 200v + 100v^2$$

with $v = 1/(1+i)$. So, v must satisfy the following quadratic:

$$100v^2 + 200v - 264.46 = 0.$$

Hence,

$$v = \frac{-200 \pm \sqrt{40000 + 105784}}{200} \approx 0.909084.$$

Finally,

$$i = \frac{1}{v} - 1 \approx 0.10.$$

Problem 1.12. (5 pts) To save for a car, Roger opens a bank account into which he initially deposits \$3,200. He deposits $\$3,500$ into the same account six months later, and \$800 three months after that. In our usual notation, assume that $i^{(2)} = 0.08$. Which is the price of the most expensive car Roger can buy at time 1 using only the money from his account?

(a) About $\$7,509$
(b) About $\$7,798$
(c) About $\$7,859$
(d) About $\$7,917$
(e) None of the above

Solution: (d)
The effective interest rate per half-year is $i^{(2)}/2 = 0.04$. The balance on his account at time 1 is

$$B = 3,200(1.04)^2 + 3,500(1.04) + 800(1.04)^{1/2} \approx 7,917.$$

Problem 1.13. (5 points) *Source: SoA Exam, May 1989, Problem #4.*
Two funds, X and Y, are started with equal deposits at time-0.

Fund X accumulates at a constant force of interest of 5%.

Fund Y is credited with a nominal rate of interest x compounded semiannually.

At the end of eight years, Fund X is 1.05 times as large as Fund Y. Find x.

(a) 0.022
(b) 0.023
(c) 0.042
(d) 0.044
(e) None of the above.

Solution: (d)
The condition on the balances in the two funds after eight years gives us

$$e^{0.05(8)} = 1.05\left(1+\frac{i^{(2)}}{2}\right)^{2(8)} \quad \Rightarrow \quad \left(1+\frac{i^{(2)}}{2}\right)^{16} \approx 1.420785 \quad \Rightarrow \quad i^{(2)} = 0.0443866.$$