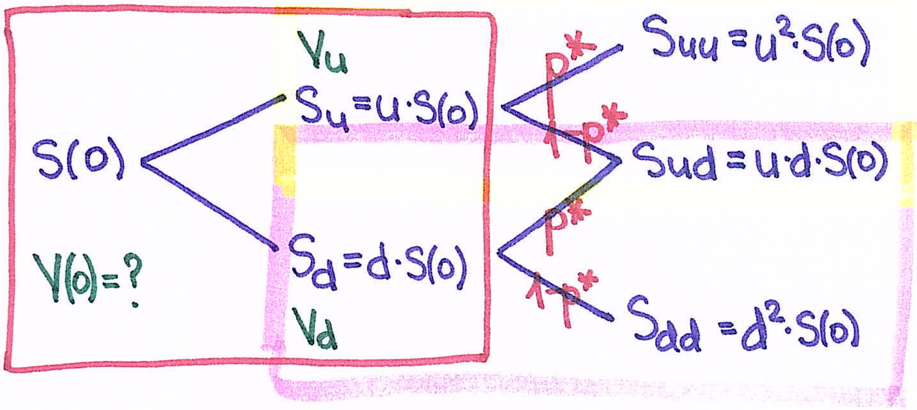
Two Binomial Periods.

04/16/2018.

Stock price tree



PAYOFF:

$V_{uu} := v(S_{uu})$

$V_{ud} := v(S_{ud})$

$V_{dd} := v(S_{dd})$

populating the tree $\longrightarrow$

$\longleftarrow$ pricing

up node: replicating portfolio @ up node

$$\begin{cases} \cdot \ \Delta_u = e^{-\delta \cdot h} \cdot \dfrac{V_{uu} - V_{ud}}{S_{uu} - S_{ud}} \\ \cdot \ B_u = e^{-r \cdot h} \dfrac{u \cdot V_{ud} - d \cdot V_{uu}}{u - d} \end{cases}$$

$\Rightarrow$ the option's value @ up node:

$$V_u = \Delta_u \cdot S_u + B_u = e^{-rh}\left[p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right]$$

w/ $p^* = \dfrac{e^{(r-\delta)\cdot h} - d}{u-d}$

down node: $\Delta_d \, ; B_d$

$$\Rightarrow V_d = \Delta_d \cdot S_d + B_d = e^{-r \cdot h}\left[p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right]$$

w/ $p^* = \dfrac{e^{(r-\delta)\cdot h} - d}{u-d}$ =

Root node:

$$V(0) = e^{-r \cdot h}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right] \quad \text{w/ } p^* \text{ as above}$$

$$= e^{-rh}\Big[p^* \cdot e^{-rh}\left(p^* \cdot V_{uu} + (1-p^*) \cdot V_{ud}\right)$$

$$+ (1-p^*) \cdot e^{-rh} \cdot \left(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd}\right)\Big]$$

$$= \underbrace{e^{-r\overbrace{(2h)}^{T}}}_{\text{discounting}} \cdot \underbrace{\left[(p^*)^2 \cdot V_{uu} + 2 \cdot p^*(1-p^*) V_{ud} + (1-p^*)^2 \cdot V_{dd}\right]}_{\text{Risk-neutral-Expected Payoff}}$$

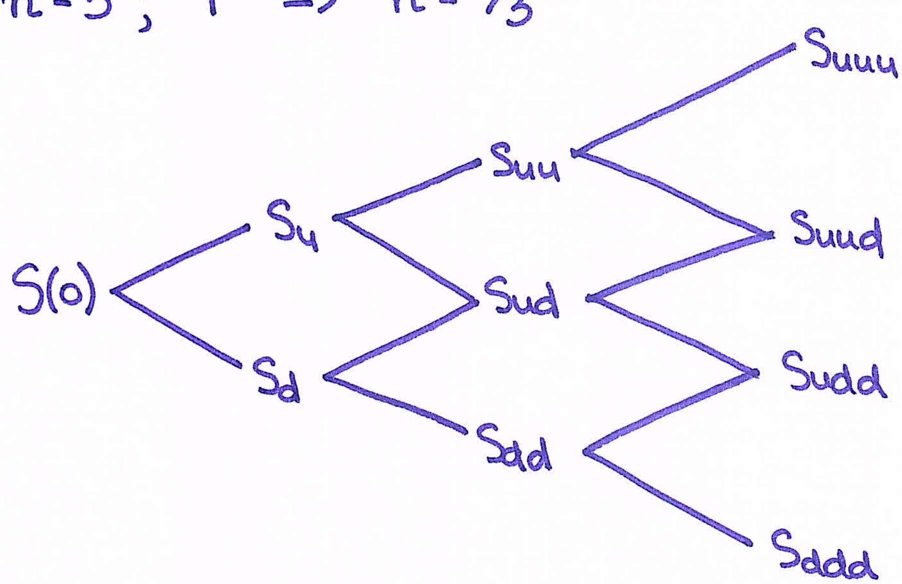
Generally (a w/out proof):

$$V(0) = e^{-r \cdot T} \cdot \underbrace{\mathbb{E}^*}\left[V(T)\right]$$

expectation under the risk-neutral measure

Three Periods

$n = 3$; T $\Rightarrow$ $h = T/3$



PAYOFF:		Risk-neutral prob:
V_{uuu}	$\times$	$(p^*)^3$
V_{uud}	$\times$	$3(p^*)^2(1-p^*)$
V_{udd}	$\times$	$3p^*(1-p^*)^2$
V_{ddd}	$\times$	$(1-p^*)^3$

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Problem Set #3

Multiple binomial periods (European).

Problem 3.1. The current price of a continuous-dividend-paying stock is $100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded, risk-free interest rate equals 0.06.

Consider a $95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

$n=3$ $T=3/4$ $\Rightarrow$ $\boxed{h=1/4}$

(a) $2.97
(b) $3.06
(c) $3.59
(d) $3.70
(e) None of the above.

forward tree:

$$u=e^{(r-\delta)h+\sigma\sqrt{h}}=e^{(0.06-0.03)\cdot\frac{1}{4}+0.2\sqrt{\frac{1}{4}}}=1.1135;$$

$$d=e^{(r-\delta)h-\sigma\sqrt{h}}=e^{0.03\cdot\frac{1}{4}-0.2\cdot\frac{1}{2}}=0.9116.$$

$S(0)$

$S_{uuu}>95$ $V_{uuu}=0$

$S_{uud}>95$ $V_{uud}=0$

$V_{ddu}=2.4665$

$S_{ddu}=S(0)\cdot d^2\cdot u=100(0.9116)^2\cdot 1.1135=92.5335$

$S_{ddd}=S(0)\cdot d^3=100(0.9116)^3=75.7553$

$V_{ddd}=19.2447$

$$p^*=\frac{1}{1+e^{\sigma\sqrt{h}}}=\frac{1}{1+e^{0.1}}=0.475$$

$$V_P(0)=e^{-0.06\cdot\frac{3}{4}}\left[19.2447(1-0.475)^3+2.4665\cdot 3\cdot 0.475\cdot(1-0.475)^2\right]$$

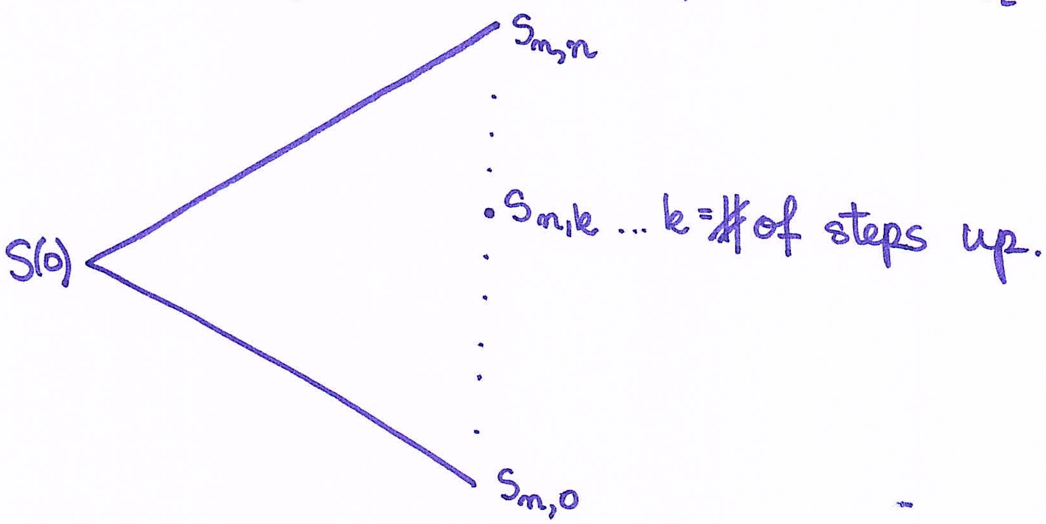
$$=3.5884$$

Multiple Binomial Periods

n ... # of periods

T ... time horizon (say, the exercise date of an option)

$\Rightarrow$ length of one period $h = \frac{T}{n}$



$S(T)$ ... a rnd variable w/ $(n+1)$ possible values

$\Rightarrow$ for every $k = 0, \ldots, n$:

$$S_{n,k} = S(0) \cdot u^k \cdot d^{n-k}$$

$\Rightarrow$ The possible payoffs of an option:

$$v_{n,k} = v(S_{n,k})$$

The risk-neutral probability of this payoff is:

$$\binom{n}{k}(p^*)^k(1-p^*)^{n-k}$$

$\Rightarrow$

$$V(0) = e^{-r \cdot T} \cdot \mathbb{E}^*[V(T)]$$

$$= e^{-rT} \cdot \sum_{k=0}^{n} \left(\binom{n}{k}(p^*)^k(1-p^*)^{n-k} \cdot v_{n,k}\right)$$

Problem 3.2. Let the continuously compounded interest rate be $r = 10\%$. Assume that the initial price of a non-dividend-paying stock is \$100 per share.

Consider a 5−period binomial model for the evolution of the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

$n = 5$ $T = 1$

(i) What is the price of a one-year, 100-strike cash call on the above asset?

$\Rightarrow h = \frac{1}{5}$

$S(0) = 100$

$S_{u^5} = S(0)u^5 = 100(1.04)^5 = 121.67$

$S_{u^4d} = S(0) \cdot u^4 \cdot d = 100(1.04)^4 \cdot (0.96) = 112.31$

$S_{u^3d^2} = S(0) \cdot u^3 d^2 = 100(1.04)^3(0.96)^2 = 103.67$

Out·o·money!