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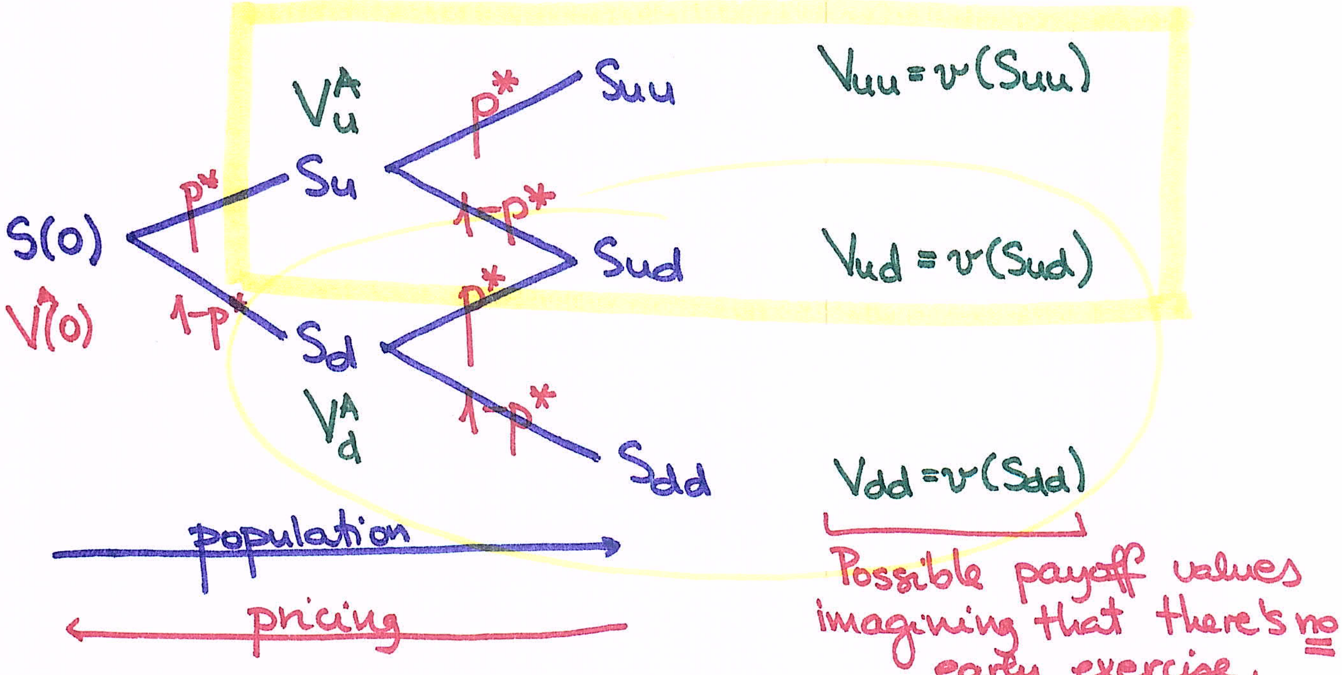
Binomial Pricing of American Options.



Consider an American option w/ payoff f'tion $v(\cdot)$

up node:
- IE_u... value of immediate exercise
- CV_u.. continuation value

↳ if we don't exercise now, the option becomes a European option (since there are no more admissible early·exercise dates left!)

$$\Rightarrow CV_u = e^{-r\cdot h}\left(p^* \cdot V_{uu} + (1-p^*)\cdot V_{ud}\right)$$

$$\Rightarrow \boxed{V_u^A = \max(IE_u, CV_u)}$$

and the option's owner decides whether to exercise early accordingly.

down node:

- IE_d
- $CV_d = e^{-rh}(p^* \cdot V_{ud} + (1-p^*) \cdot V_{dd})$

$$V_d^A = \max(IE_d, CV_d)$$

ROOT node:

- IE_0
- $CV_0 = e^{-r \cdot h}(p^* \cdot V_u^A + (1-p^*) \cdot V_d^A)$

$$\boxed{V^A(0) = \max(IE_0, CV_0)}$$

Note:

- We can still dynamically replicate American options until we reach nodes in which early exercise is optimal.
- The procedure is analogous in multiperiod trees.

$n=2$

4. For a two-period binomial model, you are given:

(i) Each period is one year. $h=1$

(ii) The current price for a nondividend-paying stock is 20. $S(0)=20$

(iii) $u=1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.

(iv) $d=0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.

(v) The continuously compounded risk-free interest rate is 5%. $r=0.05$

Calculate the price of an American call option on the stock with a strike price of 22. $K=22$

Implicitly: $T=n\cdot h=2$.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

5. Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i) The current exchange rate is 1.43 US dollars per pound.

(ii) The strike price of the put is 1.56 US dollars per pound.

(iii) The volatility of the exchange rate is $\sigma=0.3$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%.

(v) The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

(A) 0.23

(B) 0.25

(C) 0.27

(D) 0.29

(E) 0.31

$K=22$

$$\begin{tikzcd}[column sep=small]
 & & S_{uu}=32.97 & V_{uu}=10.97 \\
 & \overset{V_u^A}{S_u=25.68} \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & & \\
S(0)=20 \arrow[ur] \arrow[dr] & & S_{ud}=22.10 & V_{ud}=0.10 \\
 & S_d<K \arrow[ur, "p^*"] \arrow[dr] & & \\
 & & S_{dd}<K & V_{dd}=0
\end{tikzcd}$$

Risk-neutral Probab:

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{0.05}-0.8607}{1.284-0.8607} = 0.4502$$

up:
- $IE_u = 25.68-22 = 3.68$
- $CV_u = e^{-0.05}\left(0.4502 \cdot 10.97 + 0.5498 \cdot 0.10\right) = 4.753$

$V_u^A = 4.753 \Rightarrow$ No Early Exercise!

down: Out-o-money $\Rightarrow V_d^A = CV_d =$

$$= e^{-0.05} \cdot 0.4502 \cdot 0.10 = 0.044$$

ROOT: Out-o-money

$$\Rightarrow V^A(0) = CV_0 = e^{-0.05}\left(0.4502 \cdot 4.753 + 0.5498 \cdot 0.044\right)$$

$$= 2.0507 \Rightarrow (c) \blacksquare$$

BEGINNING OF EXAMINATION

1. You use the usual method in McDonald and the following information to construct a binomial tree for modeling the price movements of a stock. (This tree is sometimes called a forward tree.)

(i) The length of each period is one year.

(ii) The current stock price is 100.

(iii) The stock's volatility is 30%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a two-year 100-strike American call option on the stock.

(A) 11.40

(B) 12.09

(C) 12.78

(D) 13.47

(E) 14.16

1. Answer: E

We have $S_0 = 10$, $\delta = 0.05$, $\sigma = 0.3$, $r = 0.05$, and $h = 1$. By (10.10),

$$\begin{cases} u = \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 + 0.3\sqrt{1}] = e^{0.3} \\ d = \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 - 0.3\sqrt{1}] = e^{-0.3} \end{cases}.$$

By (10.5),

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0.05)\times 1} - e^{-0.3}}{e^{0.3} - e^{-0.3}} = 0.42556.$$

The stock prices and call prices are listed at each node below:

$S_0 = 100$
$C_0 = 14.16$

$S_u = 134.9859$
Early exercise: 34.9859
Continuation: 33.2796

$S_d = 74.08182$
$C_d = 0$

$S_{uu} = 182.21188$
$C_{uu} = 82.21188$

$S_{ud} = 100$
$C_{ud} = 0$

$S_{dd} = 54.8816$
$C_{dd} = 0$

For the calculation of C_u, we have

$$C_u = e^{-0.05}[p^* C_{uu} + (1 - p^*)C_{ud}] = 33.2796,$$

but early exercise would be optimal at a value of 34.9859. The time-0 price of the call is

$$C = e^{-0.05}[p^* C_u + (1 - p^*)C_d] = 14.1624.$$

Remark:
For a given volatility σ, if u and d are computed using the method of forward tree, then

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(r-\delta)h} - e^{(r-\delta)h - \sigma\sqrt{h}}}{e^{(r-\delta)h + \sigma\sqrt{h}} - e^{(r-\delta)h - \sigma\sqrt{h}}} = \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1} = \frac{1}{1 + e^{\sigma\sqrt{h}}},$$

and hence

$$1 - p^* = \frac{1}{1 + e^{-\sigma\sqrt{h}}}.$$

As a result, $p^* < \frac{1}{2}$; this provides a check for p^*.

6.

11. For a two-period binomial model for stock prices, you are given:

(i) Each period is 6 months.

(ii) The current price for a nondividend-paying stock is $70.00.

(iii) $u = 1.181$, where u is one plus the rate of capital gain on the stock per period if the price goes up.

(iv) $d = 0.890$, where d is one plus the rate of capital loss on the stock per period if the price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the current price of a one-year American put option on the stock with a strike price of $80.00.

(A) $9.75

(B) $10.15

(C) $10.35

(D) $10.75

(E) $11.05

7.

From the second equation (the gamma-neutral equation), we obtain

$$z = 65.1/0.0746 = 872.654 \approx 872.7.$$

(This is sufficient to determine that (B) is the correct answer.) Substituting this in the first equation (the delta-neutral equation) yields

$$y = 582.5 - 872.7 \times 0.7773 = -95.8.$$

11. Answer = (D)

With $u = 1.181$, $d = 0.890$, $h = 0.5$, and $\delta = 0$, the risk-neutral probability that the stock price will increase at the end of a period is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.05\times0.5} - 0.890}{1.181 - 0.890} = 0.465. \qquad (10.5)$$

For the two-period model, the stock prices are

$S_0 = 70$

$S_u = uS_0 = 1.181 \times 70 = 82.67$ $\qquad S_d = dS_0 = 0.890 \times 70 = 62.30$

$S_{uu} = uS_u = 1.181 \times 82.67 = 97.63$ $\qquad S_{ud} = dS_u = 0.890 \times 82.67 = 73.58$

$S_{dd} = dS_d = 0.890 \times 62.30 = 55.45$

Let P_0, P_u, P_d, P_{uu}, P_{ud}, P_{dd} denote the corresponding prices for the American put option. The three prices at the option expiry date are

$$P_{uu} = \max(K - S_{uu}, 0) = \max(80 - 97.63, 0) = 0,$$
$$P_{ud} = \max(K - S_{ud}, 0) = \max(80 - 73.58, 0) = 6.42,$$
$$P_{dd} = \max(K - S_{dd}, 0) = \max(80 - 55.45, 0) = 24.55.$$

By the backward induction formula (10.12), the two prices at time 1 are

$$\begin{aligned}
P_u &= \max(K - S_u, e^{-rh}[P_{uu}p^* + P_{ud}(1-p^*)]) \\
&= \max(80 - 82.67, e^{-0.05/2}[0\times0.465 + 6.42\times(1 - 0.465)]) \\
&= e^{-0.05/2}\times6.42\times0.535 \\
&= 3.35, \\
P_d &= \max(K - S_d, e^{-rh}[P_{ud}p^* + P_{dd}(1-p^*)]) \\
&= \max(80 - 62.30, e^{-0.05/2}[6.42\times0.465 + 24.55\times(1 - 0.465)]) \\
&= \max(17.70, 15.72) \\
&= 17.70.
\end{aligned}$$

Finally, the time-0 price of the American put option is

$$\begin{aligned}
P_0 &= \max(K - S_0, e^{-rh}[P_u p^* + P_d(1-p^*)]) \\
&= \max(80 - 70, e^{-0.05/2}[3.35\times0.465 + 17.70\times(1 - 0.465)]) \\
&= \max(10, 10.75) \\
&= 10.75.
\end{aligned}$$

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