APRIL 30TH

COMPOUND OPTIONS

STOCK $\leftarrow$ Option $\leftarrow$ COMPOUND Option

Option: Derivative Security

- European
- exercise date T_u $\uparrow$ for <u>underlying</u>

COMPOUND Option:

- European
- exercise date T_c

$$\boxed{T_c \leq T_u}$$

$0 \quad\quad T_c \quad\quad T_u$

$\uparrow$ The payoff of the <u>compound</u> option is paid.

$$v\big(\underbrace{V^u(T_c)}_{\text{The value of the underlying option @ time-}T_c}\big)$$

The value of the <u>underlying</u> option @ time-T_c.

v... the payoff function of the <u>compound</u> option.

Focus on the following family of options:

$\underline{\quad\quad\quad\quad}$ ON $\underline{\quad\quad\quad\quad}$

call/put   call/put

The Table of Payoffs

K_c .... strike of the compound option

Compound \ underlying	CALL	PUT
CALL	$(V_{call}(T_c)-K_c)_+$	$(V_{put}(T_c)-K_c)_+$
PUT	$(K_c-V_{call}(T_c))_+$	$(K_c-V_{put}(T_c))_+$

Put-Call Parity for compound options:

$$\text{Call·on·Call}(0) - \text{Put·on·Call}(0) = \text{Call}(0) - K_c e^{-rT_c}$$

$$\text{Call·on·Put}(0) - \text{Put·on·Put}(0) = \text{Put}(0) - K_c e^{-rT_c}$$

Consider a non-dividend-paying stock with the intial price of \$100. Assume that the continuously compounded risk-free interest rate equals 0.05.

There is an at-the-money European put option on the above stock with exercise date in 2 years. This option is currently traded at \$11.54.

A **compound call** on the above put option is issued. Its exercise date is one year from today and its strike is 6. The price of this compound call is \$7.18.

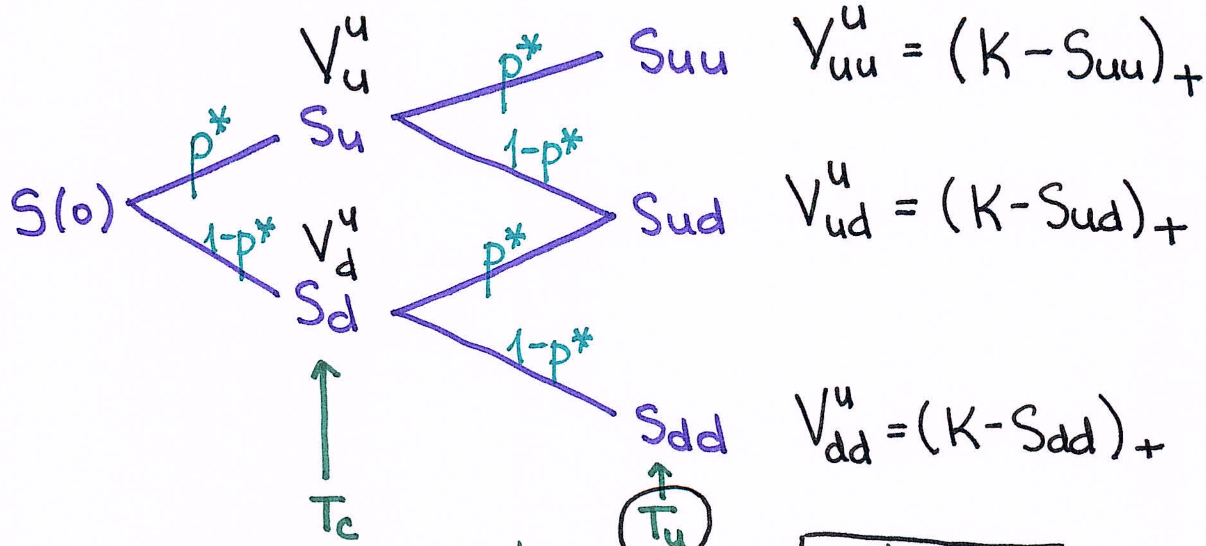
What is the value of a compound **put** option on the above vanilla option?

Put-call parity for compound options:

$$\underbrace{Call \cdot on \cdot Put(0)}_{7.18} - \underbrace{Put \cdot on \cdot Put(0)}_{?} = \underbrace{Put(0)}_{11.54} - \underbrace{K_c e^{-rT_c}}_{6e^{-0.05 \cdot 1}}$$

$$\boxed{? = 1.35}$$

The simplest binomial tree suitable for pricing compound options is a two-period tree:



Let's price a put on a put.

strike K_c — strike K

$$\left.\begin{aligned} V_u^u &= e^{-r(T_u - T_c)}\left[p^* \cdot V_{uu}^u + (1-p^*)\cdot V_{ud}^u\right] \\ V_d^u &= e^{-r(T_u - T_c)}\left[p^* \cdot V_{ud}^u + (1-p^*)\cdot V_{dd}^u\right] \end{aligned}\right\}$$

The possible payoffs of the compound option:

$$V_u = (K_c - V_u^u)_+$$

$$V_d = (K_c - V_d^u)_+$$

$$\Rightarrow \boxed{V(0) = e^{-rT_c}\left[p^* \cdot V_u + (1-p^*) V_d\right]}$$

The above approach can be generalized to:

- T_c ... the end of the k^{th} period for some $k \leq n$,
- T_u ... the end of the n^{th} (final) period in the tree.

Currency-Option pricing

- **Underlying asset** ... FOREIGN CURRENCY [FC] (an exchange rate)

  w/ r_F ... continuously compounded, risk-free interest rate for the foreign currency

- **Domestic currency** w/ [DC]

  r_D ... continuously compounded, risk-free interest rate for the domestic currency

Analogy: Foreign Currency $\longleftrightarrow$ Continuous-dividend-paying assets

$$r_F \longleftrightarrow \delta$$

Q: Does the same analogy work for binomial option pricing?

One-period x ... exchange rate

$$x(0) \xrightarrow{p^*} x_u = u \cdot x(0)$$

$$x(0) \longrightarrow x_d = d \cdot x(0)$$

$$h = T$$

PAYOFF		REPLICATING P.
$V_u = v(x_u)$	$\stackrel{?}{=}$	$\Delta e^{r_F T} \cdot x_u + B e^{r_D T}$
$V_d = v(x_d)$	$\stackrel{?}{=}$	$\Delta e^{r_F T} \cdot x_d + B e^{r_D T}$

Our **Analogy** works out!

Δ ... the number of units of FC bought @ time-0

B ... the risk-free investment in the DC

$$p^* = \frac{e^{(r_D - r_F) \cdot h} - d}{u - d}$$

In general?

Forward binomial tree :

$$\begin{cases} u = e^{(r_D - r_F)h + \sigma\sqrt{h}} \\ d = e^{(r_D - r_F)h - \sigma\sqrt{h}} \end{cases}$$

$$\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}}$$

Still : it only depends on the volatility.

SAMPLE MFE

$\$ = DC$

5. Consider a 9-month dollar-denominated American put option on British pounds.

You are given that: $T = \frac{3}{4}$ $\quad$ $£ = FC$ (underlying)

(i) The current exchange rate is 1.43 US dollars per pound. $\quad x(0) = 1.43$

(ii) The strike price of the put is 1.56 US dollars per pound. $\quad K = 1.56$

(iii) The volatility of the exchange rate is $\boxed{\sigma = 0.3}$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%. $\quad r_D = 0.08$

(v) The British pound continuously compounded risk-free interest rate is 9%. $\quad r_F = 0.09$

Using a three-period binomial model, calculate the price of the put.

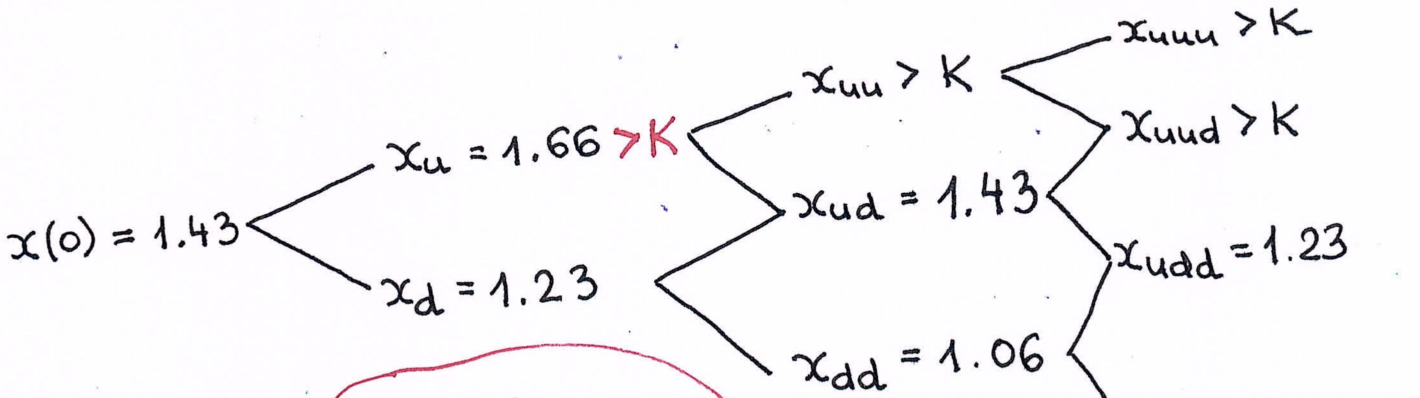
$n = 3 \Rightarrow h = \frac{1}{4}$

Q: Use the FORWARD BINOMIAL TREE (as the "default" tree)

$$u = e^{(0.08-0.09)\cdot\frac{1}{4} + 0.3\sqrt{\frac{1}{4}}} = 1.16$$

$$d = e^{-0.0025 - 0.15} = 0.86$$

The risk-neutral probability: $\boxed{p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = 0.46}$



Complete @ home!

Which tree? The default is FORWARD $\quad K = 1.56$

$$u = e^{(r_D - r_F)h + \sigma\sqrt{h}} = e^{(0.08-0.09)\cdot\frac{1}{4} + 0.3\cdot\sqrt{\frac{1}{4}}} = 1.1589$$

$$d = e^{(r_D - r_F)h - \sigma\sqrt{h}} = e^{-0.01\cdot\frac{1}{4} - 0.15} = 0.8586$$

$x(0) = 1.43$

p^* $\quad x_u = 1.66 > K$ $\qquad x_d = 1.23$

$x_{uu} > K$ $\qquad x_{ud} = 1.42$ $\qquad x_{dd} = 1.05$

$x_{uuu} > K$ $\qquad x_{uud} = 1.65 > K$ $\qquad x_{udd} = 1.22$ $\qquad x_{ddd} = 0.90$

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.15}} \approx .46$$

Finish this problem @ home!
It is Sample MFE Problem #5 :)