UNIVERSITY OF TEXAS AT AUSTIN

Lecture 13
Exchange options.

13.1. **Exchange calls.** Recall that a European call gives its owner the right, but not an obligation to

- forfeit the strike price K, and
- obtain one unit of the underlying asset in return.

We concluded that given the exercise date T, the payoff of the European call equals
Solution:

$$V_C(T) = (S(T) - K)_+.$$

More generally, we can define a derivative security which gives it owner the right, but not an obligation to

- forfeit one unit of a certain (risky) asset, and
- obtain one unit of the underlying asset in return.

In fact, a European *exchange call option* is a derivative security whose payoff depends on the values of **two** risky assets:

- the *underlying* asset (with the price temporarily denoted by $\mathbf{S} = \{S(t), t \geq 0\}$), and
- the *strike* asset (with the price temporarily denoted by $\mathbf{Q} = \{Q(t), t \geq 0\}$).

Given an exercise date T, its payoff equals
Solution:

$$V_{EC}(T, \mathbf{S}, \mathbf{Q}) = (S(T) - Q(T))_+ .$$

Note that our usual notation was modified above. The first argument is still reserved for the *valuation time*. The second argument stands for the *underlying asset*, while the third argument denotes the *strike asset*.

13.2. **Exchange puts.** Analogously, a European *exchange put option* is a derivative security whose payoff depends on the values of

- the *underlying* asset (with the price temporarily denoted by $\mathbf{S} = \{S(t), t \geq 0\}$), and
- the *strike* asset (with the price temporarily denoted by $\mathbf{Q} = \{Q(t), t \geq 0\}$).

Given an exercise date T, we obtain that its payoff equals
Solution:

$$V_{EP}(T, \mathbf{S}, \mathbf{Q}) = (Q(T) - S(T))_+ .$$

13.3. **A special symmetry.** One can immediately note the following equality between the payoffs of the exchange calls and puts:

$$V_{EP}(T, \mathbf{S}, \mathbf{Q}) = (Q(T) - S(T))_+ = V_{EC}(T, \mathbf{Q}, \mathbf{S}).$$

Due to the **law of the unique price**, it follows that for every $t \in [0, T]$

$$V_{EP}(t, \mathbf{S}, \mathbf{Q}) = V_{EC}(t, \mathbf{Q}, \mathbf{S}).$$

13.4. **Maximum options.** For two (risky) assets with prices denoted by $\mathbf{S} = \{S(t), t \geq 0\}$ and $\mathbf{Q} = \{Q(t), t \geq 0\}$ and an exercise date T, the *maximum option* is defined so that its payoff equals

$$V_{max}(T) = \max(S(T), Q(T)).$$

Problem 13.1. The financial interpretation. How would you describe the physical (financial) implementation of the maximum option? More precisely, in the case of physical settlement, which actions are the owner and the writer of the maximum option entitled to and/or obligated to?

Solution:

The **owner** of the maximum option has the right to choose one unit of either **S**, or **Q** while the **writer** has the obligation to deliver the asset of the owner's choosing.

Problem 13.2. First properties. Notice that

$$V_{max}(T) = \max(S(T), Q(T)) \geq \begin{cases} S(T) \\ Q(T) \end{cases}$$

What conclusion can you draw about the initial price of the maximum option?

Solution:

$$V_{max}(0) \geq \max(F^P_{0,T}(S), F^P_{0,T}(S))$$

Problem 13.3.

(i) Recall the following algebraic identity:

$$\max(a, b) = a + (b - a)_+ = b + (a - b)_+.$$

Rewrite the payoff of the maximum option in two distinct ways using the above identity.

Solution:

$$\begin{aligned} V_{max}(T) &= \max(S(T), Q(T)) \\ &= S(T) + (Q(T) - S(T))_+ \\ &= Q(T) + (S(T) - Q(T))_+. \end{aligned}$$

(ii) Using your response to the previous part of the problem, construct a **replicating portfolio** for the maximum option.

Solution:

The following are the four possible responses:

$\begin{cases} \text{a } \textbf{prepaid forward} \text{ on } \mathbf{Q}, \\ \text{an } \textbf{exchange call} \text{ with underlying } S \text{ and the strike asset } Q \end{cases}$

$\begin{cases} \text{a } \textbf{prepaid forward} \text{ on } \mathbf{Q}, \\ \text{an } \textbf{exchange put} \text{ with underlying } Q \text{ and the strike asset } S \end{cases}$

$\begin{cases} \text{a } \textbf{prepaid forward} \text{ on } \mathbf{S}, \\ \text{an } \textbf{exchange put} \text{ with underlying } S \text{ and the strike asset } Q \end{cases}$

$\begin{cases} \text{a } \textbf{prepaid forward} \text{ on } \mathbf{S}, \\ \text{an } \textbf{exchange call} \text{ with underlying } Q \text{ and the strike asset } S \end{cases}$