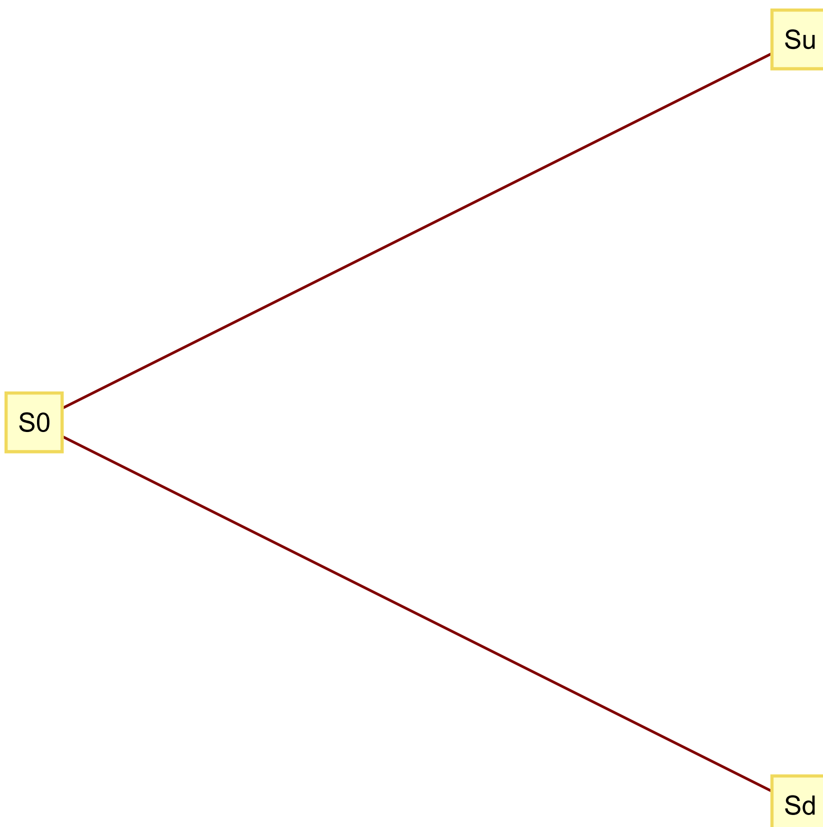
UNIVERSITY OF TEXAS AT AUSTIN

Lecture 16

The binomial asset-pricing model.

16.1. **Introduction.** We wish to introduce the simplest model for the price of a risky asset at a specific time in the future. Were we to assume that price to be deterministic, the asset itself would no longer be risky. The least number of possible values the asset price can take at time$-h$ is, of course, two. We can depict this model using a simple tree as follows:



The vertices in the above tree are called the **nodes**. The left-most node is frequently referred to as the **root node**. Its label $S(0)$ will correspond to the current (spot) asset price observed at time-0. Our modeling assumption that there are two possible asset prices at time$-h$ translate into the root node having two **offspring**: the "up" node and the "down" node. The asset price at time$-h$, denoted by $S(h)$ is, hence, a random variable with two possible values S_u and S_d. It is customary for these values to be arranged so that $S_u > S_d$. Thus, we end up with the *one-period binomial asset-price tree*. The time length h is called the length of a single period (or step) of the binomial tree.

Frequently, the model above is posited not by stipulating the values S_u and S_d, but via a pair of constants u and d. The "up" factor u and the "down" factor d satisfy the following equalities:

$$S_u = uS(0) \quad \text{and} \quad S_d = dS(0).$$

It is evident that the constants u and d are related to the possible values of the realized **simple rate of return** of the risky asset **S**. Indeed, the realized simple rate of return over the period $[0, h]$ is given by

$$\frac{S(h) - S(0)}{S(0)}.$$

Within the above model, the possible values that the realized simple rate of return can attain are:

$$\frac{S_u - S(0)}{S(0)} \quad \text{and} \quad \frac{S_d - S(0)}{S(0)}.$$

The constants u and d can be expressed as

$$u = \frac{S_u}{S(0)} = \frac{S_u - S(0)}{S(0)} + 1$$
$$d = \frac{S_d}{S(0)} = \frac{S_d - S(0)}{S(0)} + 1$$

16.2. **The no-arbitrage condition.** The model above is simplicity itself and it is, at first sight, difficult to imagine that the constants u and d could possibly be chosen poorly. However, this is an issue worth looking into. First, we need to determine what it would mean for the pair u and d to be choosen poorly. Assume that the risky asset of interest is a continuous-dividend-paying stock. The existing environment at time-0 when we build the model consists of the following:

- the continuously compounded risk-free interest rate r,
- the initial asset price $S(0)$,
- the dividend yield δ.

The binomial asset-pricing model will be ill-posed if u and d are chosen so that they introduce arbitrage in the already existing environment. The existence of an arbitrage opportunity hinges on the relationship between the risk-less and risky investment.

The risk-less investment. If the amount $S(0)$ is invested at the continuously compounded risk-free interest rate r, at the end of the investment period, the wealth of the investor will simply be equal to $S(0)e^{rh}$.

The risky investment. If the amount $S(0)$ is invested in the risky asset, the investor gets to purchase exactly one share of stock at time-0. Due to continuous immediate reinvestment of the dividend in the same stock, the number of shares of stock owned at time$-h$ equals $e^{\delta h}$. The investor's wealth is, hence, equal to $S(h)e^{\delta h}$ – a random variable. The possible values of this random variable, i.e., the possible values the wealth of the shareholder can attain are:

up: $e^{\delta h}S_u = S(0)ue^{\delta h}$ if the stock price moved "up", and

down: $e^{\delta h}S_d = S(0)de^{\delta h}$ if the stock price moved "down".

It is logical to surmise that the investor who exposes himself to risk does so because (s)he is in a sense monetarily compensated for doing so. Plainly said, the wealth resulting from the risk-less investment should be "wedged" in between the "better" and the "worse" scenario of the risky investment. Formally, we suspect the following to be true:

$$S(0)de^{\delta h} \leq S(0)e^{rh} \leq S(0)ue^{\delta h},$$

i.e.,

$$\boxed{d < e^{(r-\delta)h} < u}$$

The above is not simply a sensible assumption to be adopted at its face value. We can show that the violation of the above pair of inequalities yields an arbitrage opportunity. Assume, to the contrary, that

$$e^{(r-\delta)h} \leq d < u.$$

Based on our analysis above, we would propose the following arbitrage portfolio consisting simply of **one share of stock**. The profit of this investment equals

$$S(h)e^{\delta h} - S(0)e^{rh}.$$

So, if the stock price goes "up", the profit will satisfy

$$S_u e^{\delta h} - S(0)e^{rh} = S(0)e^{\delta h}(u - e^{(r-\delta)h}) > 0.$$

Alternatively, if the stock price goes "down", the profit will satisfy

$$S_d e^{\delta h} - S(0)e^{rh} = S(0)e^{\delta h}(d - e^{(r-\delta)h}) \geq 0.$$

Our portfolio has the profit which is always non-negative and which is strictly positive in the case that the stock price goes "up". Thus, it is indeed an arbitrage portfolio.

If one assumes the violation of the other inequality in our no-arbitrage condition, namely, if one assumes that

$$d < u \leq e^{(r-\delta)h}$$

one can construct a similar arbitrage portfolio. *An interested reader* **will** *do this.*

Remark 16.1. While it is possible to accomplish this, we will **not** consider binomial models for prices of stocks which pay discrete dividends.