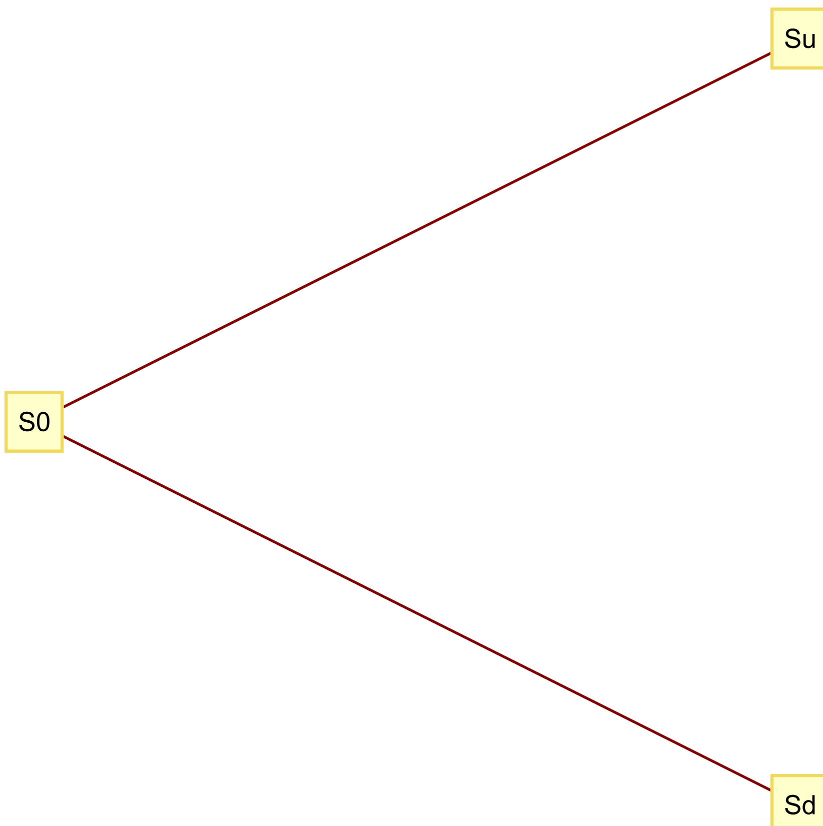
UNIVERSITY OF TEXAS AT AUSTIN

Lecture 17

Option pricing in the one-period binomial model.

17.1. **Introduction.** Recall the one-period binomial tree which we used to depict the simplest non-deterministic model for the price of an underlying asset at a future time$-h$.

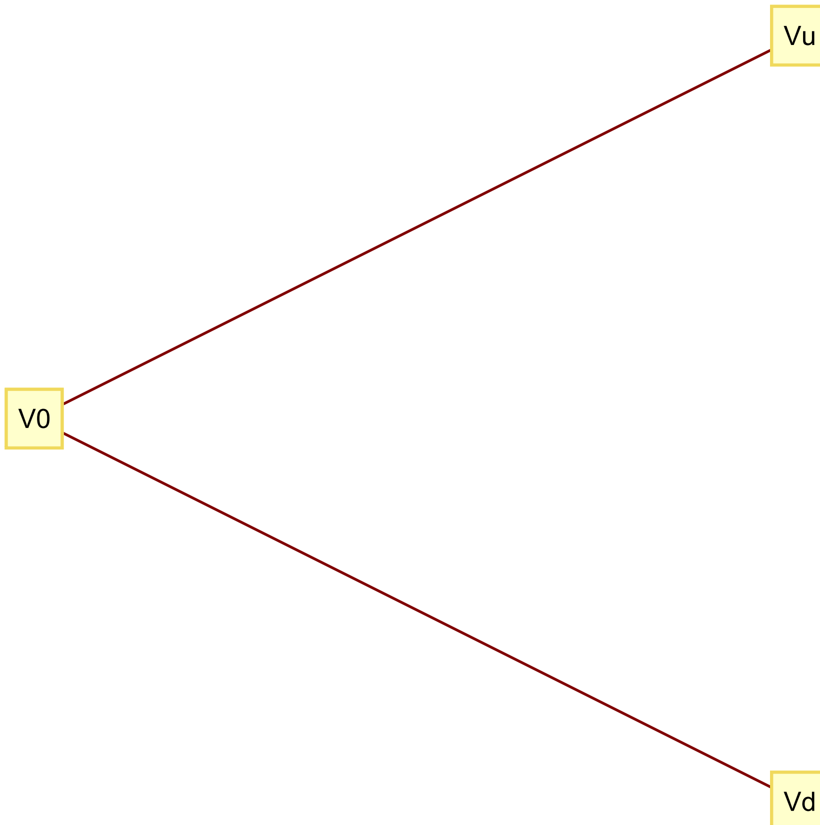


Our next objective is to determine the no-arbitrage price of a European-style derivative security with the exercise date T coinciding with the length h of our single period.

Consider such a derivative security whose payoff function is denoted by v. The payoff of this derivative security is, thus, a random variable

$$V(T) = v(S(T)) = v(S(h)).$$

Per our stock-price model above, the random variable $S(T)$ can only attain values S_u and S_d. So, the random variable $V(T)$ can only take the values $V_u := v(S_u)$ and $V_d := v(S_d)$. We can depict the resulting derivative-security tree as follows:



Note that we constructed the stock-price tree by starting from the root node containing the initial observed stock price. Then, we used our model encapsulated in the pair (u, d) of multiplicative factors to "populate" the offspring nodes. In short, we moved **from left to right**. Now that we wish to figure out the option price dictated by our stock-price tree, we start from the only known quantities: the possible payoffs. Then, we move *from right to left* to calculate the price of the derivative security occupying the root node of the derivative-security tree.

17.2. **Pricing by replication.** The method by which we intend to accomplish the above goal is the following:

Step 1. Create the replicating portfolio for our derivative security consisting of an investment in the underlying risky asset and a loan (given or taken) at the continuously compounded risk-free interest rate r.

Step 2. Calculate the initial cost of the replicating portoflio.

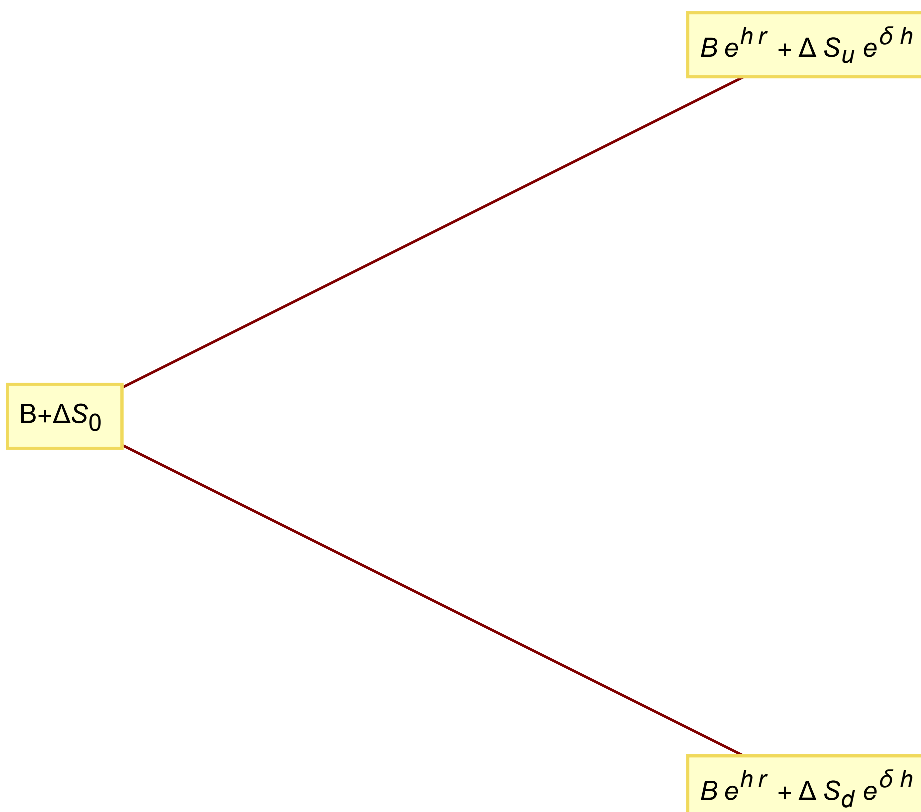
Step 3. Conclude that the no-arbitrage price $V(0)$ of our derivative security must equal the initial cost of its replicating portfolio.

Let us, again, focus on the underlying asset being a continuous-dividend-paying stock with the dividend yield δ. It is traditional to denote by Δ the initial number of units of the underlying in the replicating portoflio.[1] If $\Delta > 0$, then Δ units of the underlying asset are longed, i.e., purchased. If $\Delta < 0$, then $|\Delta|$ units of the underlying asset are shorted. In particular, in case that the underlying asset is a stock, the negative value of Δ implies a short-sale of $|\Delta|$ shares of stock. We denote the initial amount invested at the continuously compounded risk-free interest rate r by B. The choice of notation here is obvious: B stands for bonds, seeing as zero-coupon bonds are a simple device for both lending and borrowing money. If $B > 0$, then the amount B is invested at the rate r. If $B < 0$, then the amount $|B|$ is borrowed at the continuously compounded risk-free interest rate r.

Recall that in the one-period model $T = h$. With the above notation and the convention that dividends are to be continuously and immediately reinvested in the same stock, we see that the number of shares "owned" at the end of the time interval $[0, T]$ equals $\Delta e^{\delta h}$. Likewise, the riskless investment accumulated to Be^{rh}. Hence, the total value of the replicating portfolio at time$-T$ is a random variable equal to

$$\Delta e^{\delta h} S(h) + Be^{rh}\,.$$

We can depict the two possible values that the replicating portfolio can attain using the following one-period binomial tree:



In order for the above portfolio to indeed be a **replicating portfolio** of the derivative security with the payoff function v, its payoff needs to be equal to the random variable $V(T)$

[1]The significance of the notation will be discussed in M339W.

in all states of the world. Formally, we obtain the following system of equations:

$$\Delta e^{\delta h} S_u + B e^{rh} = V_u$$
$$\Delta e^{\delta h} S_d + B e^{rh} = V_d$$

If the above two equalities hold, we can conclude that the initial cost of the replicating portoflio equals the price of the derivative security, i.e.,

$$\boxed{V(0) = \Delta S(0) + B} \tag{17.1}$$

The replicating portfolio will be completely determined once we solve for Δ and B in the above system. We get

$$\Delta = e^{-\delta h}\,\frac{V_u - V_d}{S_u - S_d} \quad \text{and} \quad B = e^{-rh}\frac{uV_d - dV_u}{u-d}\,. \tag{17.2}$$

Problem 17.1. Solve for Δ and B in the above system.

17.3. **The pricing formula simplified.** The above pricing formula is already straightforward and simple. The procedure of finding Δ and B also comes in handy when we need to explicitly determine the replicating portfolio (for instance, when an arbitrage opportunity presents itself due to mispricing). However, when we merely want to calculate the price of the derivative security of interest, we can make the calculation more streamlined. Moreover, we will have a pretty nifty interpretation of the resulting (simple) pricing formula.

First, we can substitute the expressions for Δ and B from (17.2) into the pricing formula (17.1). We obtain the

$$\begin{aligned} V(0) &= \Delta S(0) + B \\ &= e^{-\delta h}\,\frac{V_u - V_d}{S_u - S_d} \times S(0) + e^{-rh}\frac{uV_d - dV_u}{u-d} \\ &= e^{-\delta h}\,\frac{V_u - V_d}{S(0)(u-d)} \times S(0) + e^{-rh}\frac{uV_d - dV_u}{u-d} \\ &= e^{-rh}\left[\frac{e^{(r-\delta)h} - d}{u-d} \times V_u + \frac{u - e^{(r-\delta)h}}{u-d} \times V_d\right]. \end{aligned} \tag{17.3}$$

If the numerators of the coefficients next to V_u and V_d look familiar, this is rightfully so. We have seen bits and pieces of those expressions in the **no-arbitrage condition** for the binomial asset-pricing model. In fact, we can conclude that both of the coefficients are **non-negative** and that they **sum up to one**. In other words, the weighted sum of the two possible payoffs is actually a **convex combination** of the two possible payoffs. In fact, the weights in the above convex combination can be interpreted as **probabilities**.

Definition 17.1. The *risk-neutral probability* of the asset price moving up in a single step in the binomial tree is defined as

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

Remark 17.2. The probability measure $\mathbb{P}^*$ giving the probability p^* to the event of moving up in a single step and the probability $1 - p^*$ to the event of moving down in a single step in the binomial tree is called the *risk-neutral probability measure.* This probability measure and the rationale for its name will be discussed in M339W.

Combining Definition 17.1 with the result of calculations in (17.3), we get the following **risk-neutral pricing formula**:

$$\boxed{V(0) = e^{-rT}[p^* V_u + (1-p^*)V_d]} \tag{17.4}$$

It is customary to interpret (and memorize) the above formula by noting that the initial value of the derivative security is equal to its discounted expected payoff under the risk-neutral probability measure. We even write

$$V(0) = e^{-rT}\mathbb{E}^*[V(T)]$$

where $\mathbb{E}^*$ denotes the expectation associated with the risk-neutral probability measure $\mathbb{P}^*$.

Problem 17.2. MFE Exam, Spring 2007: Problem #14

For a one-year straddle on a non-dividend-paying stock, you are given:

- The straddle can only be exercised at the end of one year.
- The payoff of the straddle is the absolute value of the difference between the strike price and the stock price at expiration date.
- The stock currently sells for \$60.00.
- The continuously compounded risk-free interest rate is 8%.
- In one year, the stock will either sell for \$70.00 or \$45.00.
- The option has a strike price of \$50.00.

Calculate the current price of the straddle.

(A) \$0.90
(B) \$4.80
(C) \$9.30
(D) \$14.80
(E) \$15.70

Solution: Our intention is to use the risk-neutral pricing formula (17.4). The length of our one time-period is one year, so $h = T = 1$. The stock pays no dividends, so that $\delta = 0$. With the remaining data explicitly provided in the problem statement, we get that the risk-neutral probability of the stock price going up equals

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{S(0)e^{(r-\delta)h} - S_d}{S_u - S_d} = \frac{60e^{0.08} - 45}{70 - 45} \approx 0.8.$$

The two possible payoffs of the straddle are

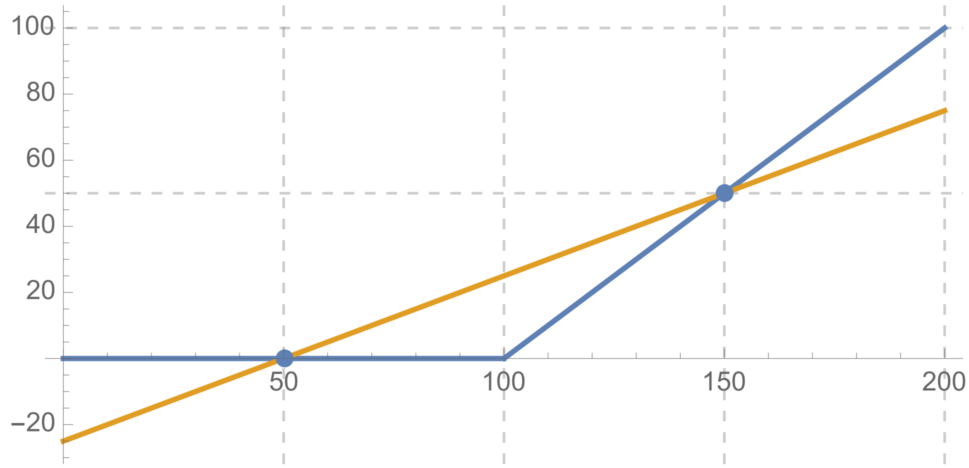
$$V_u = |S_u - K| = |70 - 50| = 20 \quad \text{and} \quad V_d = |45 - 50| = 5.$$

Finally, we obtain

$$V(0) = e^{-0.08}[0.8 \times 20 + 0.2 \times 5] \approx 15.693.$$

We choose the offered choice $\boxed{\text{E.}}$

17.4. **Graphical interpretation of binomial pricing of call and put options.** For simplicity, let us assume that the stock does not pay dividends in this example. The following image contains the payoff curve of the call option (the blue curve) on the same coordinate system as the payoff curve of its replicating portfolio (the orange line).



We see that the two graphs intersect precisely at the two asset prices which are possible in the associated one-period binomial model. These are the only two points at which the two curves coincide. This is sufficient to claim that the replicating portfolio is, indeed, replicating for the call in the given model since the two possible stock prices signify the only two possible states-of-the-world.

The positive slope of the orange line indicates that the Δ in the replicating portfolio is itself positive. This means that the replicating portfolio for a call will **always** entail **purchasing** shares of stock. Moreover, the slope is necessarily smaller than one, so that the number of purchased shares of stock is always less than one.

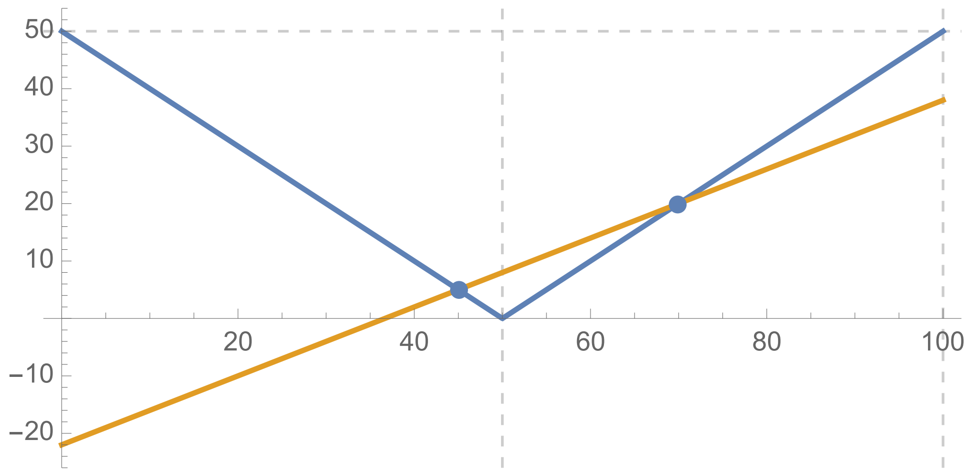
The negative vertical-axis intercept of the orange line implies that $B < 0$. Thus, borrowing $|B|$ will always be part of the replicating portfolio for a call option.

Similarly, one can obtain that the Δ of the put option will always be between -1 and 0. So, the replicating portfolio for a put will necessarily entail short-selling the underlying. The B in the put's replicating portfolio will always be non-negative. Thus, the amount B is initially deposited to earn the continuously compounded risk-free interest rate r.

Problem 17.3. Consider the straddle in Problem 17.2.

- Draw the payoff curve of the straddle.
- In the same coordinate system, draw the payoff curve of the straddle's replicating portfolio in the given one-period binomial model for the stock price.
- Find the Δ and the B for the replicating portfolio.
- Precisely describe the investment in the risky asset and the risk-less investment in the replicating portfolio.

Solution:



The replicating portfolio consists of:

- $\Delta = 3/5 \Rightarrow 3/5$ purchased shares of stock;
- $B = -22e^{-0.08} \Rightarrow$ borrowed $\$22e^{-0.08}$.

17.5. **Arbitrage opportunities due to mispricing in the one-period binomial model.** The source of an arbitrage opprotunity in this setting will be an observed option price which is inconsistent with the no-arbitrage price as dictated by the binomial asset-pricing model. To investigate how one can exploit such an arbitrage opportunity, let us look into a problem first.

Problem 17.4. MFE Exam, Spring 2009: Problem #3

You are given the following regarding stock of Widget World Wide (WWW):

- The stock is currently selling for \$50.
- One year from now the stock will sell for either \$40 or \$55.
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 10%.
- The continuously compounded risk-free interest rate is 5%.

While reading the Financial Post, Michael notices that a one-year at-the-money European call written on stock WWW is selling for \$1.90. Michael wonders whether this call is fairly priced. He uses the binomial option pricing model to determine if an arbitrage opportunity exists. What transactions should Michael enter into to exploit the arbitrage opportunity (if one exists)?

(A) No arbitrage opportunity exists.
(B) Short shares of WWW, lend at the risk-free rate, and buy the call priced at \$1.90.
(C) Buy shares of WWW, borrow at the risk-free rate, and buy the call priced at \$1.90.
(D) Buy shares of WWW, borrow at the risk-free rate, and short the call priced at \$1.90.
(E) Short shares of WWW, borrow at the risk-free rate, and short the call priced at \$1.90.

Qualitative analysis of offered answers.

(A) It would strike one as quite unlikely that the SoA would pose an arbitrage problem in which there is no arbitrage opportunity. So, one would be inclined to discard this option.
(B) No obvious shortcomings in this answer!

(C) Simultanously longing both the shares of stock and the call option cannot eliminate risk. So, we discard this offered answer.
(D) No obvious shortcomings in this answer!
(E) Simultaneously short-selling the underlying and writing the call option cannot eliminate risk. So, we discard this offered answer.

The conclusion is that a cursory investigation of the offered answer allows one to increase the probability of guessing correctly if pressed for time! In the present problem, one would toss a mental coin to decide between (B) and (D).

Solution: Although the offered answers are just sketches of potential arbitrage portfolio. This problem can serve as a template for all similar problems we may encounter in the future. So, let us solve it in a tad more detail than necessary.

Diagnosis. Since we ultimately want to construct an arbitrage portfolio, it makes sense to immediately find the Δ and B and use them for pricing. In general, we have

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S_u - S_d}$$

In this problem, the length of the period is one year so that $h = 1$. The two possible payoffs are $V_u = 5$ and $V_d = 0$. So, we get

$$\Delta = e^{-0.1} \times \frac{5}{55-40} \approx 0.3016.$$

As for the risk-free investment, we have

$$B = e^{-0.05} \times \frac{55 \times 0 - 40 \times 5}{55-40} \approx -12.6831.$$

So, the no-arbitrage call price

$$V(0) = \Delta S(0) + B = 0.3016 \times 50 - 12.6831 = 2.3969.$$

Since $V(0) \neq 1.9$, we conclude that there is, indeed, an arbitrage opportunity.

Construction. Since the no-arbitrage price exceeds the observed price, we conclude that the observed call option is **underpriced**. So, an arbitrage portfolio must include a **purchase** of the observed call option. *At this point in the exam, you would choose* $\boxed{B.}$ *and move on!*

For didactic purposes, let us completely construct the arbitrage portfolio to consist of the following components:

- one **long** observed call option,
- **short-sale** of Δ shares of stock,
- a **deposit** of $-B$ to earn the continuously compounded risk-free interest rate r.

The latter two components combined can be described a the **short replicating portfolio** of the call option.

Verification. The inital cost of our proposed arbitrage portfolio equals

$$1.90 - V(0) < 0,$$

meaning that there is an initial inflow of funds. As for the payoff, note that the Δ and B were chosen exactly so as to create the replicating portfolio, so the payoff of the total proposed arbitrage portfolio is by design equal to zero. We conclude that we have, indeed, created an arbitrage portfolio.

Problem 17.5. With the data and the model from Problem 17.4, find the no-arbitrage call price using the risk-neutral pricing formula.

Solution: The risk-neutral probability equals

$$p^* = \frac{50e^{0.05-0.10} - 40}{55 - 40} \approx 0.5041.$$

So, the call's price equals

$$V_C(0) = e^{-0.05} \times 5 \times 0.5041 = 2.3976.$$

The following flowchart contains the steps one needs to take in order to exploit an arbitrage opportunity arising from mispricing in the one-period binomial model. Let us denote the observed price of a certain derivative security by χ, and let us denote its **no-arbitrage** price dictated by a one-period binomial model for the price of the underlying asset by $V(0)$. The number of shares of stock in the replicating portfolio for our derivative security is denoted by Δ and the amount invested at the continuously compounded risk-free interest rate by B.