Prepaid-forward & forward prices (cont'd)

Case #3. Discrete dividend-paying stocks.

Prepaid Forward Contract

\[ F_{0,T}^P(S) = ? \]

\[ F_{0,T}^P(S) = S(0) - \sum_{k=1}^{m} PV_{0,t_k}(D_k) \]

Forward Contract

\[ F_{0,T}(S) = FV_{0,T}(F_{0,T}^P(S)) \]

\[ \Rightarrow F_{0,T}(S) = FV_{0,T}(S(0)) - \sum_{k=1}^{m} FV_{t_k,T}(D_k) \]
11.1. **Construction.** So far, we have looked at put-call parity for non-dividend-paying assets. Now, we will use a similar approach to obtain put-call parity for stocks that pay either discrete dividends, or a continuous dividend stream.

Let **Portfolio A** consist of a long European call and a short European put on the same underlying asset $S$ with the same strike $K$ and the same exercise date $T$. The initial value of this portfolio is

$$V_A(0) = V_C(0) - V_P(0).$$

There are no intermediate cash-flows associated with this portfolio and its payoff at time $T$ is

$$V_C(T) - V_P(T) = S(T) - K. \quad (A)$$

On the other hand, let **Portfolio B** consist of the following:

1. a long prepaid forward contract on $S$ for delivery at time $T$,
2. borrowing the present value of the strike price to be repaid at time $T$.

Then, the initial cost of this portfolio equals:

$$F_{0,T}^P(S) - PV_{0,T}(K).$$

Since there are no intermediate cash-flows associated with this portfolio, either, its payoff at time $T$ is

$$S(T) - K. \quad (B)$$

Since the above portfolios have the same final payoff, by the no-arbitrage principle, we conclude that their initial values must also be the same. We get the more general version of put-call parity:

$$V_C(0) - V_P(0) = F_{0,T}^P(S) - PV_{0,T}(K).$$

11.2. **Special cases.** Our most common setting is the one with a continuously compounded interest rate $r$. In that case the put-call parity reads as

$$V_C(0) - V_P(0) = F_{0,T}^P(S) - Ke^{-rT}.$$  

With respect to dividends, these are the three cases we will be looking into:

- non-dividend-paying stocks:
  $$V_C(0) - V_P(0) = S(0) - Ke^{-rT}$$
- discrete dividends $D_i, i = 1, \ldots, n$ at times $0 < t_1 < \cdots < t_n \leq T$:
  $$V_C(0) - V_P(0) = S(0) - \sum_{i=1}^{n} D_i e^{-rt_i} - Ke^{-rT}$$
Payoff diagram for Portfolio A:

\[ V_A(T) = V_c(T) - V_p(T) \]
\[ = (s(T) - K)_+ - (K - s(T))_+ \]
\[ = \begin{cases} 
  s(T) - K & \text{if } s(T) \geq K \\
  -(K - s(T)) & \text{if } s(T) < K 
\end{cases} \]

\[ V_A(T) = s(T) - K \]
bullet continuous dividends at the rate $\delta$:

$$V_C(0) - V_P(0) = S(0)e^{-rT} - Ke^{-rT}$$

**Problem 11.1. MFE Exam Spring 2007: Problem #1**

On April 30, 2007, a common stock is priced at $52.00. You are given that:

1. Dividends in equal amounts are to be paid on June 30, 2007, and on September 30, 2007.
2. A European call on the above stock with strike $K = 50$ and the exercise date in six months sells for $4.50$.
3. A European put on the above stock with strike $K = 50$ and the exercise date in six months sells for $2.45$.
4. The continuously-compounded risk-free interest rate equals 0.06.

Calculate the amount of each dividend.

**Solution:** In addition to our usual notation, we introduce $D$ to stand for the amount of each dividend payment. Then, the put-call parity reads as

$$V_C(0) - V_P(0) = S(0) - De^{-rt_1} - De^{-rt_2} - Ke^{-rT}$$

with $t_1 = 1/6$ and $t_2 = 5/12$. Solving for $D$ above, we get

$$D = \frac{S(0) - Ke^{-rT} - V_C(0) + V_P(0)}{e^{-rt_1} + e^{-rt_2}} = \frac{52 - 50e^{-0.06(1/2)} - 4.5 + 2.45}{e^{-0.06(1/6)} + e^{-0.06(5/12)}} \approx 0.73.$$
Put-call Parity

\[ V_c(0) - V_p(0) = \text{F}_{0,T}^P(S) - \text{PV}_{0,T}(K) \]

In this problem, there are discrete dividends:

\[ V_c(0) - V_p(0) = S(0) - D e^{-rt_1} - D e^{-rt_2} - Ke^{-rT} \]

\[ 4.50 - 2.45 = 52 - D e^{-0.06 \cdot \frac{1}{6}} - D e^{-0.06 \cdot \frac{5}{12}} - 50e^{-0.06 \cdot \frac{1}{2}} \]

\[ D = 0.73 \]
"Synthetic" Forwards

Payoff

Long Call + Short Put = "synthetic" forward

Long call

s(final asset price)

K

Short Put

-K
Def'n. Consider a European-Style derivative security w/ payoff $V(T)$. A portfolio is said to be a **REPLICATING PORTFOLIO** for this derivative security if it has the same payoff.

**Note**: The above is the "static" version.

Q.: Can you create a replicating portfolio for a forward contract using a call and a put on the same underlying?

\[
\begin{align*}
\text{Long call} & \quad \text{w/ strike price equal to } K = F_{0,T}(S) \\
\text{Short put} & \quad \\
\end{align*}
\]

• What is the initial cost of this replicating portfolio? Should be zero as there is no initial cost to the forward contract. Indeed,

\[
V_C(0) - V_P(0) = F_{0,T}^p(S) - PV_{0,T}(F_{0,T}(S)) = 0
\]

Put-call Parity

**Note**: Replicating portfolios need not be unique. For instance:

• Long a prepaid forward contract

• Borrow $F_{0,T}^p(S)$ @ the risk-free rate to be repaid on the delivery date $T$. 
Note. Replicating portfolios are useful in pricing (ergo in exploiting arbitrage as well). Given a derivative security to price we:

- Construct a replicating portfolio consisting of "simpler" ingredients.
- Price the replicating portfolio.

[Think about Lunchables as an example.]