Law of the unique price
Assume NO ARBITRAGE.

Let Portfolios A and B be:
  * static and
  * $V_A(T) = V_B(T)$

Conclusions?

$V_A(0) = V_B(0)$

Proof. Assume, to the contrary, that $V_A(0) \neq V_B(0)$.}

Without loss of generality:

- $V_A(0) < V_B(0)$
  - relatively CHEAP
  - Buy/LONG Portfolio A
  - relatively EXPENSIVE
  - "Sell"/SHORT Portfolio B

The payoffs match in all states of the world, i.e., for all final risky-asset prices.

Law of the unique price

I. DIAGNOSIS

II. PROPOSAL of an arbitrage portfolio
2. **TOTAL PORTFOLIO**

- **Long Portfolio A**
- **Short Portfolio B**

### III. VERIFICATION

Q: Is this really an ARBITRAGE PORTFOLIO?

In general: \( \text{PROFIT} = \text{PAYOFF} - \text{FV(INIT. COST)} \).

- **Initial Cost (Total Portfolio)**: \( V_A(0) - V_B(0) < 0 \)

\[ \Rightarrow \]

Initial Inflow of money. ⚫

- **Payoff (Total Portfolio)**: \( V_A(T) - V_B(T) = 0 \)

\[ \Rightarrow \]

Profit (Total Portfolio) = \( 0 - \text{FV}_{0,T}(V_A(0) - V_B(0)) > 0 \)

\[ < 0 \]

(STRONG) ARBITRAGE.
**Forward Price.**

Basic ways of acquiring stock:

<table>
<thead>
<tr>
<th>Cash</th>
<th>Asset</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>0</td>
<td>T</td>
</tr>
</tbody>
</table>

1. Outright Purchase
2. Fully-leveraged Purchase
3. Forward Contract
4. Prepaid Forward Contract

The prepaid-forward price is paid

The relationship between the forward price & the prepaid-forward price.

Prepaid forward:

\[ F_P \]

Forward:

\[ F \]

Savings acct/bond:

\[ PV - (F \times T) \]
Payoff (A) = \( S(T) \)
Payoff (B) = \( \underbrace{S(T) - F}_{\text{forward}} + F = S(T)} \)

\[ \begin{align*}
\text{Equal} \\
\downarrow \\
F_{0,T}^p = PV_{0,T}(F_{0,T}) \\
\iff F_{0,T} = FV_{0,T}(F_{0,T})
\end{align*} \]

Forward / Prepaid - forward prices (STOCKS).

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Case #1. Non-dividend-paying stocks.

Prepaid forward
\[ \downarrow F^p \]
\[ 0 \to T \]

Outright purchase
\[ \downarrow S(0) \]
\[ \uparrow \text{unit of asset} \]

\[ \text{PAYOFF} = S(T) \]

\[ \Rightarrow F_{0,T}^p(S) = S(0) \]

\[ \Rightarrow F_{0,T}(S) = FV_{0,T}(S(0)) \]
Problem 7.1. Sample FM(DM) #18.

You are a jeweler who buys gold, which is the primary input needed for your products. One ounce of gold can be used to produce one unit of jewelry. Assume that the cost of all other inputs is negligible. You are able to sell each unit of jewelry for 700 plus 20\% of the market price of gold in one year. In one year, the actual price of gold will be in 1 of 3 possible states, corresponding to the following probability table:

<table>
<thead>
<tr>
<th>Market Price of Gold in 1 year</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = 750$ per ounce</td>
<td>$p_1 = 0.2$</td>
</tr>
<tr>
<td>$x_2 = 850$ per ounce</td>
<td>$p_2 = 0.5$</td>
</tr>
<tr>
<td>$x_3 = 950$ per ounce</td>
<td>$p_3 = 0.3$</td>
</tr>
</tbody>
</table>

You are considering utilizing forward contracts to lock in 1-year gold prices, in which case you would charge the customer (one year from now) 700 plus 20\% of the forward price. The 1-year forward price for gold is 850 per ounce. How much does your expected 1-year profit, per unit of jewelry sold, increase if you buy forward the 1-year price of gold?

A. 0
B. 8
C. 12
D. 20
E. 32

Problem 7.2. Sample FM(DM) Problem #7

A non-dividend paying stock currently sells for 100. One year from now the stock sells for 110. The risk-free rate, compounded continuously, is 6\%. The stock is purchased in the following manner:

1. You pay 100 today
2. You take possession of the security in one year.

Which of the following describes this arrangement?

\times A. Outright purchase
\times B. Fully leveraged purchase
\check C. Prepaid forward contract
\times D. Forward contract
\check E. This arrangement is not possible due to arbitrage opportunities

\textcolor{red}{\text{TRAP!}}

\textcolor{red}{\text{Environment}}

\textcolor{red}{\text{Given } F^p = 100 : \text{ the only possible source of an arbitrage opportunity.}}

\textcolor{red}{\text{We derived to no-arbitrage } F_{0,T}^p (S) = g(s) = 100}$
Case #2. Continuous-dividend-paying stock. 

\[ \delta \ldots \text{dividend yield} \]

Convention: Immediate/Continuous reinvestment of dividend in the same asset:

\[ \Rightarrow \text{shareholder does not experience any intermediate cashflows} \Rightarrow \text{STATIC PORTFOLIO} \]

\[ \Rightarrow \text{the # shares owned @ time} - t: \]

\[ N(t) = N(0) \cdot e^{\delta t} \]

Prepaid forward:

- \[ F^P_0 \]
- \[ 0 \]
- \[ T \]
- \[ \uparrow \text{unit of asset} \]

Outright purchase:

- \[ 0 \]
- \[ T \]
- \[ \downarrow e^{-\delta T} S(0) \]
- \[ \# \text{of shares owned is 1} \]
- \[ \downarrow \]
- \[ \# \text{of shares we need to buy at time} - 0: e^{-\delta T} \]

By the law of the unique price:

\[ F^P_{0,T} (S) = S(0) e^{-\delta T} \]

continuously compounded risk-free rate

\[ F_{0,T} (S) = e^{rT - \delta T} S(0) = e^{(r-\delta)T} S(0) \]
Problem 7.3. The current price of a stock is $S(0) = \$1.25$ per share. Let the stock pay continuous dividends at the continuous dividend rate $\delta$. Assume that the continuously compounded interest rate equals $r = 0.3$. The prepaid forward price for delivery of the above stock in two years is $\$83.79$. Calculate the annualized forward premium (rate).

Problem 7.4. Suppose that the current price of a dividend-paying stock equals $\$1,000$. Let $r = 0.25$ and $\delta = 0.15$. You notice that a forward price for delivery of this stock in two-years equals $F = \$1,200$. You suspect that this forward price creates an arbitrage opportunity.

1. Propose an arbitrage portfolio.
2. Verify that the portfolio you proposed is indeed an arbitrage portfolio.

I. **Diagnosis**

- **Calculate the no-arbitrage forward price** $F_{0,T}(S)$.
- See if it matches the **observed** $F = \$1,200$.

$$F_{0,T}(S) = e^{(0.25 - 0.15)T} \cdot S = e^{0.1 \cdot 2} \cdot 1,000 = 1,000e^{0.2} > 1,200 = F$$

$\neq \Rightarrow$ There exists an arbitrage portfolio.

II. **Propose** an arbitrage portfolio.

$$F_{0,T}(S) > F$$

**relatively** relatively expensive **relatively** cheap