Prepaid Swaps

Note: Compare them to old-fashioned magazine subscriptions.

\[ \text{pmt \ atm} = ? \]

\[ x^p \ldots \text{prepaid - swap price} \]

PREPAID SWAP \( \iff \) \{ A portfolio consisting of two prepaid forward contracts w/ delivery dates \( T_1 \) and \( T_2 \).

An example of a REPLICATING PORTFOLIO.

\[ \implies x^p = F_{0,T_1}^p + F_{0,T_2}^p \]

NO ARBITRAGE \( \uparrow \)

\[ x^p = \frac{F_{0,T_1}}{(1+r_0(T_1))^T_1} + \frac{F_{0,T_2}}{(1+r_0(T_2))^T_2} \]

Example. Need for oil delivery @ \( T_1 = 1 \) and \( T_2 = 2 \).

Forward prices for oil delivery are \( F_{0,1} = \$20 \) and \( F_{0,2} = \$21 \) per barrel.

The spot rates for \( 1 \text{ yr} \) and \( 2 \text{ yrs} \) to maturity are 0.06 and 0.065, resp.

Find the prepaid - swap price.
Commodity swaps [cont'd]

\[ x^P = \frac{F_{0,T_1}}{(1 + r_0(o,T_1))^T_1} + \frac{F_{0,T_2}}{(1 + r_0(o,T_2))^T_2} \]

... Level swap price

No arbitrage \( \Rightarrow \) \( x^P = PV_{0,T_1}(x) + PV_{0,T_2}(x) \)

\( \Rightarrow \) \( x^P = \frac{x}{(1 + r_0(0,T_1))^T_1} + \frac{x}{(1 + r_0(0,T_2))^T_2} \)

\( \Rightarrow \) \( x = \frac{x^P}{(1 + r_0(0,T_1))^{-T_1} + (1 + r_0(0,T_2))^{-T_2}} \)

Example [oil, cont'd] Level swap price = ?

\( x = 37.383 \left( (1.06)^{-1} + (1.065)^{-2} \right)^{-1} = \ldots = 20.483 \).

Note:

\( F_{0,1} = 20 < x = 20.483 < F_{0,2} = 21 \)

We always (i.e., in the normal cases):

\( F_{0,1} \leq x \leq F_{0,2} \)
Example [cont'd]

\[ F_{0,1} = 20 \quad F_{0,2} = 21 \]
\[ r_0(0,1) = 0.06 \quad r_0(0,2) = 0.065 \]

\[ \Rightarrow [x = 20.483] \]

level swap:

pair of forwards:

\[ \begin{align*}
& x & x \\
& V & \wedge \\
& F_{0,1} & F_{0,2}
\end{align*} \]

It appears that we "overpay" at time-1 and we "underpay" at time-2

\[ \Rightarrow \text{Interpretation:} \]

This acts as a "loan" from the buyer to the seller:

- issued @ time-1 \{ the correct
- repaid @ time-2 \} rate to be charged

is the IMPLIED FORWARD RATE for \([1,2]\).

\[ r_0(T_1,T_2) \]

In our example:

\[ (1+r_0(0,1)) (1+r_0(1,2)) = (1+r_0(0,2))^2 \]

\[ r_0(1,2) = \left( \frac{1.065}{1.06} \right)^2 - 1 = 0.07 \]

Check if the interpretation works:

\[ (x - F_{0,1}) (1 + r_0(1,2)) = F_{0,2} - x \]
The Market Value of a Swap

- The market value of a swap is initially zero
- Once the swap is struck, its market value will generally no longer be zero because
  - the forward prices for oil and interest rates will change over time
  - even if prices do not change, the market value of swaps will change over time due to the implicit borrowing and lending
- A buyer wishing to exit the swap could enter into an offsetting swap with the original counterparty or whomever offers the best price
- The market value of the swap is the difference in the PV of payments between the original and new swap rates
Let \( P_1 \) and \( P_2 \) represent the one-year and two-year forward prices per ton of rice, respectively.

Let \( r_1 \) and \( r_2 \) represent the one-year and two-year spot rates, respectively.

A rice buyer and a rice supplier agree that the supplier will deliver one ton of rice at the end of each of the next two years, and the buyer will pay a constant swap price of \( P \) per ton.

Determine an expression for \( P \).

(A) \( \frac{P_1 + P_2}{2} \)

(B) \( \frac{r_1 P_1 + r_2 P_2}{r_1 + r_2} \)

(C) \( \frac{(P_1 + P_2)(1 + r_1)(1 + r_2)^2}{1 + r_1 + (1 + r_2)^2} \)

(D) \( \frac{P_1(1 + r_2)^2 + P_2(1 + r_1)}{1 + r_1 + (1 + r_2)^2} \)

(E) \( \frac{P_1(1 + r_1) + P_2(1 + r_2)^2}{1 + r_1 + (1 + r_2)^2} \)

\[
\frac{P_1}{1 + r_1} + \frac{P_2}{(1 + r_2)^2} = \frac{P}{1 + r_1} + \frac{P}{(1 + r_2)^2} \]

\[
P = \frac{P_1 (1 + r_2)^2 + P_2 (1 + r_1)}{1 + r_1 + (1 + r_2)^2} \]

\[
P \left( (1 + r_1) + (1 + r_2)^2 \right) = P
\]

\[
P = \frac{P_1 (1 + r_2)^2 + P_2 (1 + r_1)}{1 + r_1 + (1 + r_2)^2}
\]

\[
\Rightarrow \boxed{\text{D}}
\]
Futures.

A good source: HULL: "Options, futures and other derivative securities" [p. 18 onwards]

- Credit risk: A MARGIN ACCOUNT
- Marking-to-market

Standardization

$\Rightarrow$ Futures are liquid

$\Rightarrow$ Confident in the observed prices.

$\Rightarrow$ Futures are suitable as underlying assets for other option.
Margin accts and marking-to-market

NOTIONAL VALUE... the initial worth of the entire investment

MARGIN ACCT... earns interest

\[ L \text{ in Full Generality this could be a non-deterministic, time-varying interest rates, i.e. @ FLOATING RATE} \]

\[ L \text{ IN OUR PROBLEMS:} \]

\[ \text{a single, risk-free, deterministic} \]

INITIAL MARGIN:  \[ B^b(0) = B^s(0) = q \cdot N \]

\[ \uparrow \text{the notional value} \]

\[ \uparrow \text{the required percentage} \]

Settlement times

\[ 0 \quad t_1 \quad \ldots \quad t_k \quad \ldots \quad T = t_n \]

\[ B^s(t_{k+1}) \]

\[ t_{k-1} \quad t_k \]

Just prior to marking to market @ time-\( t_k \):

\[ B^s(t_k^-) = B^s(t_{k-1}) \cdot e^{r(t_k - t_{k-1})} \]

\[ B^s(t_k) = B^s(t_k^-) \cdot n \times \text{Size} \times \left( F_{t_k,T} - F_{t_{k-1},T} \right) \]

\[ \text{marking to market } \text{MM} \]

the maintenance margin (the "broker's comfort boundary")
\[ B^s(t^+_k) = B^s(t_k) \] MM

The possible margin call.