Name:

M339D=M389D Introduction to Actuarial Financial Mathematics
University of Texas at Austin
Solution: The Prerequisite In-Term Exam
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 90 points.
Time: 50 minutes

TRUE/FALSE

0.1 (2)	TRUE	FALSE
0.2 (2)	TRUE	FALSE
0.3 (2)	TRUE	FALSE
0.4 (2)	TRUE	FALSE
0.5 (2)	TRUE	FALSE

MULTIPLE CHOICE

0.11 (5)	a	b	c	d	e
0.12 (5)	a	b	c	d	e
0.13 (5)	a	b	c	d	e
0.14 (5)	a	b	c	d	e
0.15 (5)	a	b	c	d	e
0.16 (5)	a	b	c	d	e
0.17 (5)	a	b	c	d	e
0.18 (5)	a	b	c	d	e
0.19 (5)	a	b	c	d	e
0.20 (5)	a	b	c	d	e

FOR THE GRADER'S USE ONLY:

T/F	F.R.	M.C.	Σ

0.1. **TRUE/FALSE QUESTIONS.** *Please, circle the correct answer on the front page of this exam.*

Problem 0.1. (2 points)

We define the minimum of two values in the usual way, i.e.,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x \geq y \end{cases}$$

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$x - \min(x - y, 0) = \max(x, y)$$

True or false?

Solution: TRUE

$$\begin{aligned} x - \min(x - y, 0) &= \begin{cases} x - 0 = x, & \text{if } x \geq y \\ x - (x - y) = y, & \text{if } x < y \end{cases} \\ &= \max(x, y) \end{aligned}$$

Problem 0.2. (2 points) Let $x > 0$. Then, we always have $e^x > 1 + x$. *True or false?*

Solution: TRUE

Problem 0.3. (2 points) Denote the continuously compounded, risk-free interest rate by r and denote the equivalent annual effective interest rate by i. Then, $\ln(1 + i) = r$. *True or false?*

Solution: TRUE

Problem 0.4. (2 pts) Two dice are rolled, the single most probable sum of the numbers of the upturned faces is 7. *True or false?*

Solution: TRUE

Problem 0.5. (2 pts)

We define the maximum of two values in the usual way, i.e.,

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x \leq y \end{cases}$$

Then, for every x and y we have that

$$\max(x, y) = \max(x - y, 0) + y$$

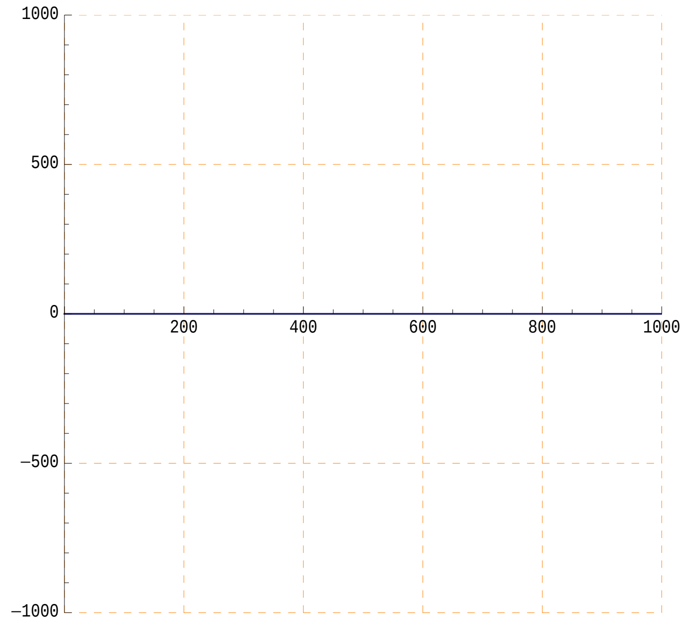
True or false?

Solution: TRUE

0.2. FREE-RESPONSE PROBLEMS.

Problem 0.6. (5 points) Draw the graph of the following function in the coordinate system provided below:

$$f(x) = \begin{cases} 300 & \text{for } 0 < x < 200 \\ 500 - x & \text{for } 200 \le x < 500 \\ 0 & \text{for } x \ge 500 \end{cases}$$



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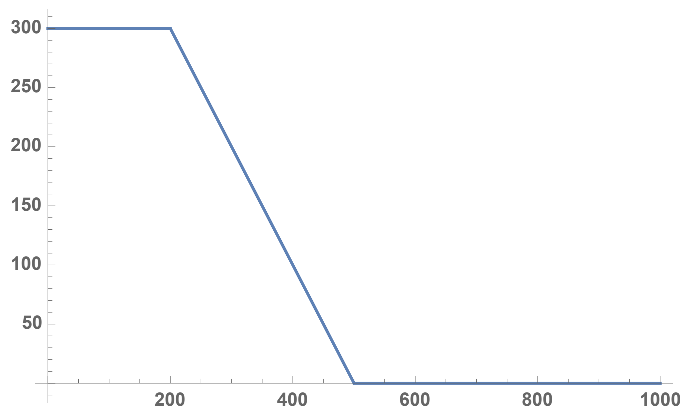
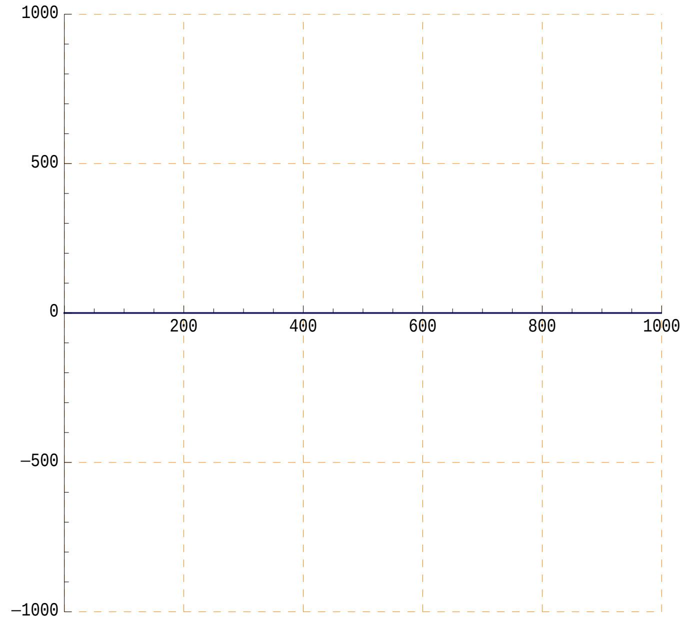


Problem 0.7. (6 points) Draw the graph of the following function in the coordinate system provided below:

$$f(x) = \begin{cases} |x - 300| & \text{for } x < 500 \\ 700 - x & \text{for } x \geq 500 \end{cases}$$



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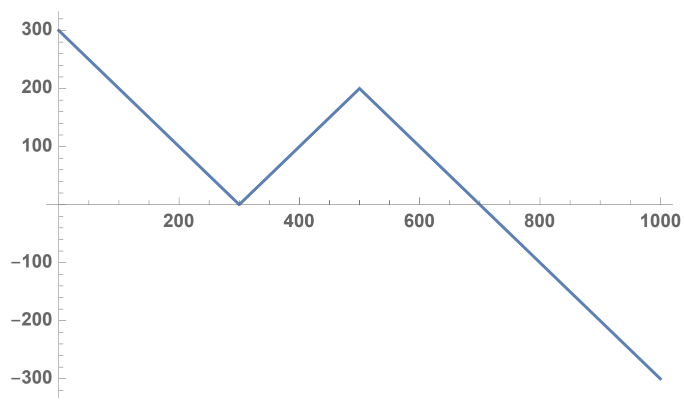
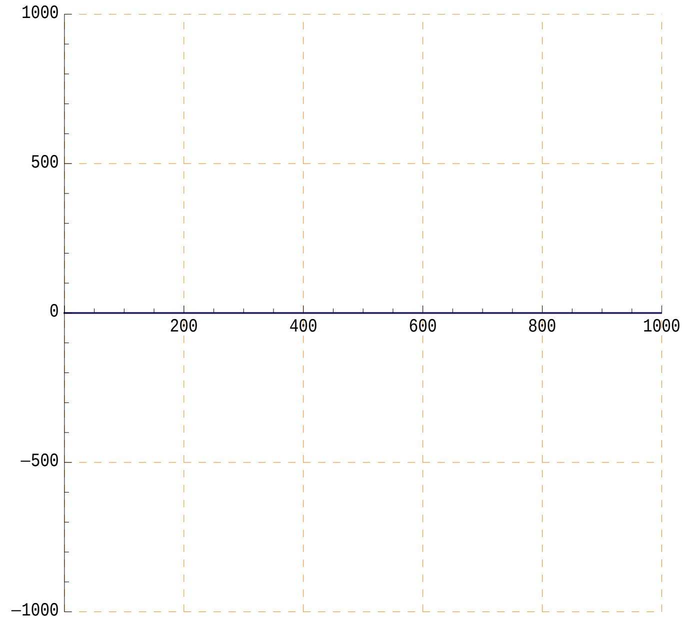


Problem 0.8. (6 points) Draw the graph of the following function in the coordinate system provided below:

$$f(x) = \begin{cases} |x - 400| & \text{for } x < 500 \\ 2x - 400 & \text{for } x \geq 500 \end{cases}$$



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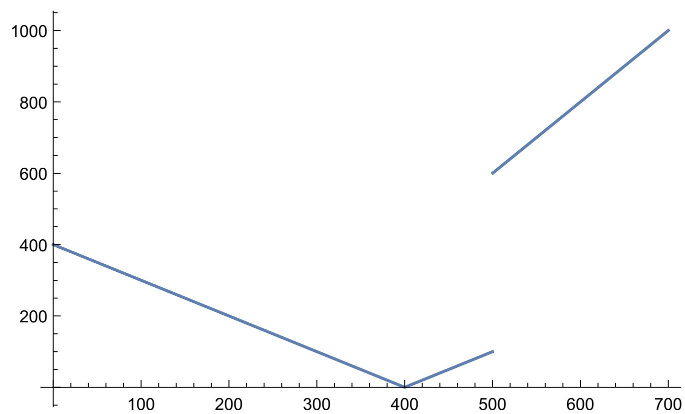


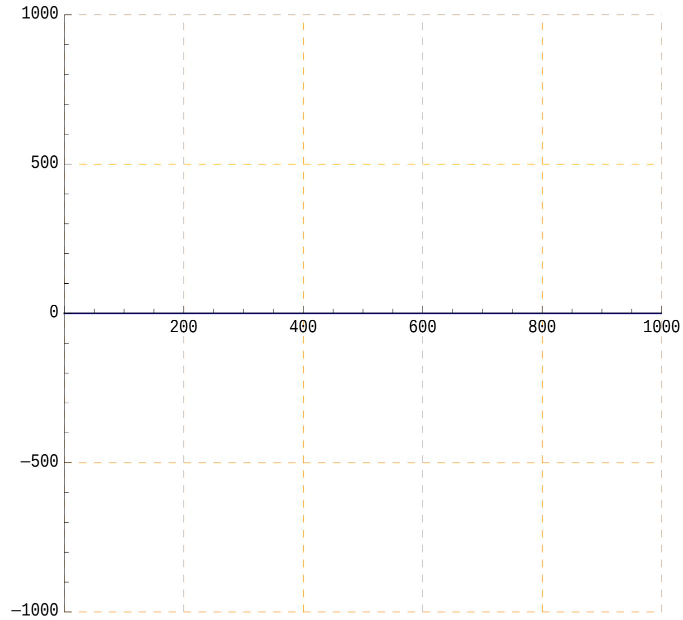
Problem 0.9. (6 points) Let the function f be given by

$$f(x) = \begin{cases} x - 300 & \text{for } x \geq 300 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graph of the function g defined as

$$g(x) = f(x) - 50$$

in the coordinate system provided.



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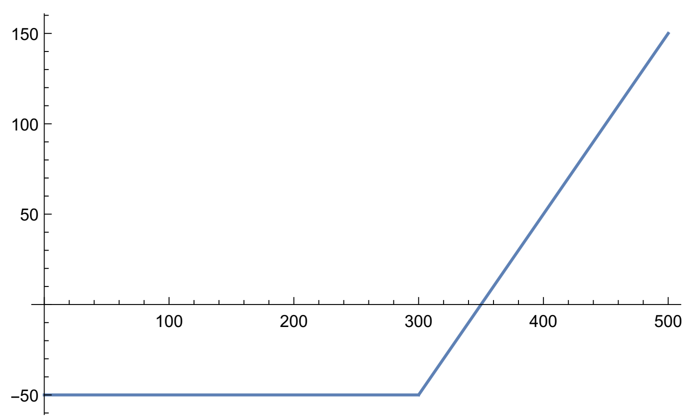


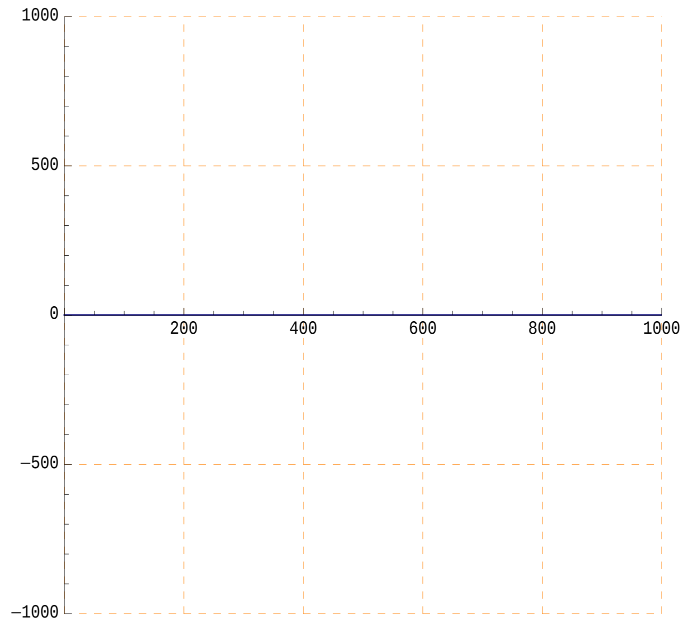
Problem 0.10. (8 points) Let the function f be defined as

$$f(x) = x$$

Let the function g be defined as

$$g(x) = \begin{cases} 0 & \text{for } x < 500 \\ x - 500 & \text{for } x \geq 500 \end{cases}$$

Draw the graph of the function $f - g$ in the coordinate system provided below:



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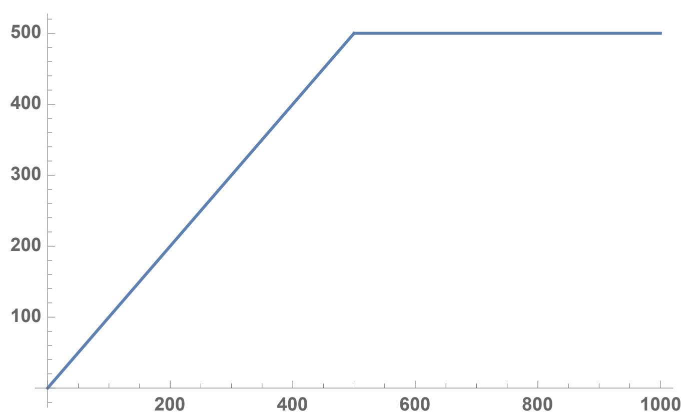


0.3. MULTIPLE CHOICE QUESTIONS.

Please, circle the correct answer on the front page of this exam.

Problem 0.11. (5 pts) A coin for which *Heads* is twice as likely as *Tails* is tossed twice, independently. If the outcomes are two consecutive *Heads*, Bertie gets a \$15 reward; if the outcomes are different, Bertie gets a \$10 reward; if the outcomes are two consecutive *Tails*, Bertie has to pay \$5. What is the expected value of Bertie's winnings?

(a) 75/9
(b) 80/9
(c) 85/9
(d) 95/9
(e) None of the above.

Solution: (d)
Since *Heads* is twice as likely as *Tails*, *Heads* appears with probability 2/3, while *Tails* appears with probability 1/3.

Let X denote the amount Bertie wins. Then, X has the following distribution:

$$X \sim \begin{cases} 15, & \text{with probability } 4/9, \\ 10, & \text{with probability } 4/9, \\ -5, & \text{with probability } 1/9. \end{cases}$$

$$\mathbb{E}[X] = \frac{4}{9}(15) + \frac{4}{9}(10) + \frac{1}{9}(-5) = \frac{95}{9}\,.$$

Problem 0.12. (5 pts) Which of the following formulas hold for the exponential function:

(a) $\frac{1}{1+e^x} = \frac{1-e^{-x}}{e^x - e^{-x}}$
(b) $(e^x)^y = e^{xy}$
(c) $e^{x+y} = e^x e^y$
(d) All of the above.
(e) None of the above.

Solution: The correct answer is (d).

Problem 0.13. (5 pts) Find the probability of obtaining exactly two *fives* in three rolls of a fair die, **given** that there is exactly one *five* in the first two rolls.

(a) $5/(2 \cdot 3)$
(b) $5/(2^2 \cdot 3)$
(c) $5/6^2$
(d) $1/6$
(e) None of the above

Solution: (d)

Problem 0.14. (5 pts) Let Y be a random variable such that $\mathbb{P}[Y = 2] = 1/2$, $\mathbb{P}[Y = 3] = 1/3$ and $\mathbb{P}[Y = 6] = 1/6$. Then $\mathbb{E}[\min(Y, 5)] = \dots$

(a) 2

(b) 17/6

(c) 3

(d) 19/6

(e) None of the above.

Solution: The correct answer is **(b)**.

$$\mathbb{E}[\min(Y, 5)] = \frac{1}{2}(2) + \frac{1}{3}(3) + \frac{1}{6}(5) = \frac{17}{6}\,.$$

Problem 0.15. (5 pts) A 5-year loan for 10,000 is charged an effective interest rate of 6% per half-year period.
The loan is to be repaid so that interest is repaid at the end of every 6 month period as it accrues and the principal is repaid in total at the end of the 5 years.

Denote the total amount of interest paid on this loan by I. Then,

(a) $I \approx 2,750$
(b) $I \approx 3,000$
(c) $I \approx 3,250$
(d) $I \approx 3,500$
(e) None of the above

Solution: (e)

$$10 \cdot (0.06) \cdot 10,000 = 6,000.$$

Problem 0.16. (5 pts) Let $\Omega = \{a_1, a_2, a_3, a_4\}$ be an outcome space, and let $\mathbb{P}$ be a probability distribution on Ω. Assume that $\mathbb{P}[\{a_1, a_2\}] = 1/3, \mathbb{P}[\{a_2, a_3\}] = 1/4$ and $\mathbb{P}[\{a_1, a_3\}] = 1/9$. Then we have that $\mathbb{P}[\{a_4\}]$ equals the following value:

(a) 1/4
(b) 11/18
(c) 7/36
(d) 47/72
(e) None of the above

Solution: (d)
For any outcome space Ω, from the axioms of probability, we must have that $\mathbb{P}[\Omega] = 1$. In this case, $\Omega = \{a_1, a_2, a_3, a_4\}$, and so

$$\mathbb{P}[\Omega] = \mathbb{P}[\{a_1, a_2, a_3, a_4\}] = \mathbb{P}[\{a_1, a_2, a_3\}] + \mathbb{P}[\{a_4\}] = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{9}\right) + \mathbb{P}[\{a_4\}].$$

Hence,

$$\mathbb{P}[\{a_4\}] = 1 - \frac{25}{72} = \frac{47}{72}.$$

Problem 0.17. (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions given by

$$f(x) = 2x - 10$$

and

$$g(x) = \begin{cases} \min(x, 7) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, $g(f(7))$ equals ...

(a) -4
(b) 0
(c) 4

(d) 7

(e) None of the above

Solution: (c)

Problem 0.18. (5 pts) Tuppy deposits \$100 into an account at time 0.
For the following three years and three months, he does not make any subsequent withdrawals or deposits and the account earns at a constant continuously compounded, risk-free interest rate r.

In the end, the balance in his account equals \$114. Then,

(a) $0 \leq r < 0.0150$
(b) $0.0150 \leq r < 0.0250$
(c) $0.0250 \leq r < 0.0550$
(d) $0.0550 \leq r < 0.0650$
(e) None of the above

Solution: (c)
The unknown continuously compounded, risk-free interest rate r must satisfy

$$114 = 100e^{3.25r}.$$

So,

$$r = \ln(1.14)/3.25 \approx 0.0403.$$

Problem 0.19. (5 points) Let the accumulation function be given by

$$a(t) = (1 + 0.05)^{t^2}$$

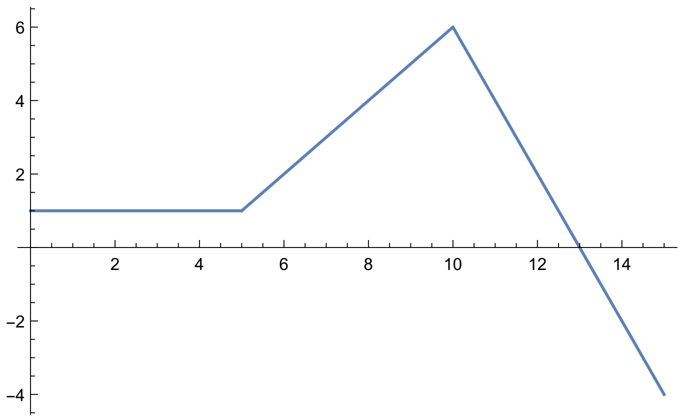
Then, we can say the following about the continuously compounded, risk-free interest rate r associated with the above accumulation function:

(a) $r = 0.05$
(b) $r = 2\ln(1.05)$
(c) $r = 0.10$
(d) The continuously compounded, risk-free interest rate is not constant.
(e) None of the above

Solution: (d)

$$r = \frac{d}{dt}\ln(a(t)) = \frac{d}{dt}\ln[1.05^{t^2}] = 2t\ln(1.05).$$

Problem 0.20. (5 points) Consider the following graph of a function $f : [0, \infty) \to \mathbb{R}$.



Then, we can say that the function f is ...

(a) increasing.

(b) decreasing.

(c) both increasing and decreasing.

(d) neither increasing nor decreasing.

(e) None of the above

Solution: (d)