Warm-up Worksheet # 3

Convexity.

Please, provide a complete solution to the following problem(s):

**Problem 3.1.** (2 points)
Let \( x_L \leq x^* \leq x_R \). Express \( x^* \) as a convex combination of \( x_L \) and \( x_R \). More precisely, find \( \lambda \) such that
\[
x^* = \lambda x_L + (1 - \lambda) x_R.
\]

**Definition 3.1.** A function \( f : [0, \infty) \to \mathbb{R} \) is said to be convex if for every \( x_L < x_R \) and every \( 0 \leq \lambda \leq 1 \) we have
\[
f(\lambda x_L + (1 - \lambda) x_R) \leq \lambda f(x_L) + (1 - \lambda) f(x_R)
\]

**Problem 3.2.** (5 points)
Provide an example of a convex function \( f : [0, \infty) \to \mathbb{R} \) (by drawing its graph in a coordinate system or by providing the explicit expression for the function).
Problem 3.3. (8 points)
The increasing function $f : [0, \infty) \rightarrow \mathbb{R}$ satisfies the following inequality for every choice of $x_1 < x_2 < x_3$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Show that the function $f$ is convex.