M339J: February 14th, 2020.

Problem. [VARIANCE OF A TWO-POINT MIXTURE]

$$X \sim \begin{cases} X_1 \sim Gamma(\alpha=2, \theta=4) & \text{w/ probab. } 0.2 \\ X_2 \sim Gamma(\alpha=2, \theta=6) & \text{w/ probab. } 0.8 \end{cases}$$

Find the variance of X.

→: Usually, we use the computational formula:

$$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

In general, for a k-pt mixture X of $X_1, X_2, \dots, X_k$ we have that

$$(\star) \quad \mathbb{E}[X^\ell] = a_1 \cdot \mathbb{E}[X_1^\ell] + a_2 \cdot \mathbb{E}[X_2^\ell] + \dots + a_k \cdot \mathbb{E}[X_k^\ell]$$

- For discrete: Use the insight:

$$p_X(j) = a_1 \cdot p_{X_1}(j) + a_2\, p_{X_2}(j) + \dots + a_k \cdot p_{X_k}(j)$$

- For continuous: By def'n of the k-pt mixture, we have

$$F_X(x) = a_1 \cdot F_{X_1}(x) + a_2 \cdot F_{X_2}(x) + \dots + a_k \cdot F_{X_k}(x)$$

Differentiating w/ respect to x:

$$(*) \qquad f_X(x) = a_1 \cdot f_{X_1}(x) + a_2 \cdot f_{X_2}(x) + \dots + a_k \cdot f_{X_k}(x)$$

The pdf of the k-pt mixture is the convex combination of the pdfs of the dist'n from which it is constructed.

Now, multiply $(*)$ by x^ℓ, and get

$$\int_{-\infty}^{\infty} x^\ell f_X(x)\,dx = a_1 \int_{-\infty}^{\infty} x^\ell f_{X_1}(x)\,dx + \dots + a_k \int_{-\infty}^{\infty} x^\ell f_{X_k}(x)\,dx$$

Using the def'n of the expectation, we see that ✪ is true.

- <u>Mixed</u>: Just combine the two arguments we used above!

<u>CAVEAT</u>: $\mathrm{Var}[X] \neq a_1 \mathrm{Var}[X_1] + \dots + a_k \mathrm{Var}[X_k]$

<u>In our problem</u>:

$$\mathbb{E}[X_1] = \theta_1 \cdot \alpha_1 = 4 \cdot 2 = 8$$

$$\mathbb{E}[X_2] = \theta_2 \cdot \alpha_2 = 6 \cdot 2 = 12$$

$$\Rightarrow \mathbb{E}[X] = 0.2 \cdot 8 + 0.8 \cdot 12 = \frac{56}{5} = 11.2$$

$$\mathbb{E}[X_1^2] = \theta_1^2(\alpha_1+1)\cdot\alpha_1 = 4^2\cdot 3\cdot 2 = 96,$$

$$\mathbb{E}[X_2^2] = \theta_2^2(\alpha_2+1)\cdot\alpha_2 = 6^2\cdot 3\cdot 2 = 216.$$

$$\Rightarrow \quad \mathbb{E}[X^2] = 0.2(96) + (0.8)216 = 192$$

$$\Rightarrow \mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= 192 - (11.2)^2 = 66.56$$

Assume that the losses X for year y are modeled using an exponential distribution with mean 200.

Due to projected inflation of 5%, we model the losses for the following year $y+1$ as $Z = 1.05X$.

(i) Find the probability that a year y loss is greater than 400.

(ii) Find the probability that a year $y+1$ loss is greater than 400.

$X \sim Exp(\text{mean} = \theta = 200)$

(i) $\mathbb{P}[X > 400] = e^{-\frac{400}{200}} = e^{-2} \approx 0.135;$

(ii) $\mathbb{P}[Z > 400] = e^{-\frac{400}{210}} \approx 0.149$

Since exponential is scale, we know

$Z \sim Exp(\text{mean} = 1.05 \cdot 200 = 210)$

Raising to a Power.

Theorem. Let X be a continuous random variable w/ density function $f_X(\cdot)$. Let its cumulative dist'n f'tion be such that $F_X(0)=0$.

Let $\tau \neq 0$ be a constant & set

$$Y = X^{1/\tau}.$$

Then, • for $\tau > 0$:

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X^{1/\tau} \leq y]$$
$$= \mathbb{P}[X \leq y^\tau] = F_X(y^\tau)$$

and

$$f_Y(y) = \tau \cdot y^{\tau-1} \cdot f_X(y^\tau) \quad y>0.$$

• for $\tau < 0$:

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X^{1/\tau} \leq y]$$
$$= \mathbb{P}[X \geq y^\tau] = 1 - F_X(y^\tau)$$

and

$$f_Y(y) = -\tau \cdot y^{\tau-1} \cdot f_X(y^\tau) \quad y>0.$$

Example. Start w/ $X \sim Exp(mean = \theta)$.

Define $Y = X^{-1}$ (i.e., $\tau = -1$)

→: For $y \geq 0$:

the c.d.f. of the random var Y:

$$F_Y(y) = 1 - F_X(y^{-1}) = S_X\left(\frac{1}{y}\right)$$

$$= e^{-\frac{\frac{1}{y}}{\theta}} = e^{-\frac{1}{y \cdot \theta}}$$

$\Rightarrow$ $Y \sim InvExp\left(\theta' = \frac{1}{\theta}\right)$. ■

Example. Let $X \sim Exp(mean = \theta)$.

Set $Y = X^{1/\tau}$ w/ $\tau > 0$

→: For $y > 0$:

$$F_Y(y) = F_X(y^\tau) = 1 - e^{-\frac{y^\tau}{\theta}}$$

$$= 1 - e^{-\frac{y^\tau}{(\theta^{1/\tau})^\tau}} = 1 - e^{-\left(\frac{y}{\theta^{1/\tau}}\right)^\tau}$$

$\Rightarrow$ $Y \sim Weibull(\theta' = \theta^{1/\tau}, \tau)$ ■

Exponentiation

X ... continuous w/ any support and density $f_X(\cdot)$ such that $f_X(x) > 0$

Let $Y = e^X$

Then, $F_Y(y) = F_X(\ln(y))$;

$$f_Y(y) = \frac{1}{y} f_X(\ln(y)) \qquad y > 0$$