M339J: February 19th, 2020.

Problem. Assume that

$$X \mid \Lambda = \lambda \sim \text{Exponential}(\text{mean} = \theta = \lambda)$$

w/ $\boxed{\Lambda \sim U(50,100)}$.

Find the (unconditional) coefficient of variation of X.

→: The coefficient of variation: $\frac{\sigma_X}{\mu_X}$.

· Focus on $\mu_X = \mathbb{E}[X]$.

We know: $\mathbb{E}[X] = \mathbb{E}[\underbrace{\mathbb{E}[X \mid \Lambda]}_{\text{ALWAYS, a function of } \Lambda}]$

In this problem:

$$\mathbb{E}[X \mid \Lambda] = \Lambda$$

conditionally exponential :)

$$\Rightarrow \mathbb{E}[X] = \mathbb{E}[\Lambda] = 75.$$

1.

· Focus on $\mathrm{Var}[X] = \sigma_X^2$.

We know:

$$\boxed{\mathrm{Var}[X] = \mathrm{Var}[\underbrace{\mathbb{E}[X \mid \Lambda]}_{\Lambda}] + \mathbb{E}[\underbrace{\mathrm{Var}[X \mid \Lambda]}_{\Lambda^2}]}$$

$$\mathrm{Var}[\underbrace{\Lambda}_{\sim U(50,100)}] = \mathrm{Var}[\cancel{50} + \underbrace{(\Lambda - 50)}_{\sim U(0,50)}] = \mathrm{Var}[50 \cdot \underbrace{\frac{\Lambda - 50}{50}}_{\sim U(0,1)}]$$

$$= 50^2 \cdot \frac{1}{12} = \frac{2500}{12}$$

$$\mathbb{E}[\Lambda^2] = \mathrm{Var}[\Lambda] + (\mathbb{E}[\Lambda])^2$$

$$= \frac{2500}{12} + (75)^2$$

$$\Rightarrow \mathrm{Var}[X] = \frac{2500}{12} + \frac{2500}{12} + (75)^2 = 6{,}042.$$

$$\Rightarrow \sigma_X = 78$$

$\Rightarrow$ Our coeff. of variation : $\frac{78}{75} \approx 1.04$

PROBLEM. Let X be a mixture dist'n w/ the mixing variable Λ such that

$$X \mid \Lambda = \lambda \sim \text{Exponential}\left(\text{mean} = \frac{1}{\lambda}\right)$$

$$\Lambda \sim \text{Exponential}(\text{mean} = 5).$$

Find the unconditional $\mathbb{P}[X \leq 3] = F_X(3)$.

→: Since X is a mixture dist'n, we can express its c.d.f. as

for $x > 0$:

$$F_X(x) = \int_0^{+\infty} F_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda)\, d\lambda$$

Using the specific dist'ns given in the problem, we have:

$$\begin{cases} \cdot\ f_\Lambda(\lambda) = \frac{1}{\theta} e^{-\frac{\lambda}{\theta}} & \lambda > 0 \\ \cdot\ F_{X|\Lambda}(x|\lambda) = 1 - e^{-\frac{x}{\frac{1}{\lambda}}} = 1 - e^{-\lambda x} \end{cases}$$

So:

$$F_X(x) = \int_0^{+\infty} (1 - e^{-\lambda x}) \cdot \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}\, d\lambda$$

$$= \underbrace{\int_0^{+\infty} \frac{1}{\theta} e^{-\frac{\lambda}{\theta}}\, d\lambda}_{=1} - \frac{1}{\theta} \int_0^{+\infty} e^{-\lambda x} \cdot e^{-\frac{\lambda}{\theta}}\, d\lambda$$

$$F_X(x) = 1 - \frac{1}{\theta}\int_0^{+\infty} e^{-\lambda\left(x+\frac{1}{\theta}\right)}\, d\lambda$$

$$= 1 + \frac{1}{\theta}\cdot\frac{1}{x+\frac{1}{\theta}}\, e^{-\lambda\left(x+\frac{1}{\theta}\right)}\Big|_{\lambda=0}^{+\infty}$$

$$= 1 + \frac{\frac{1}{\theta}}{x+\frac{1}{\theta}}(0-1)$$

$$= 1 - \frac{\frac{1}{\theta}}{x+\frac{1}{\theta}}$$

$$\Rightarrow \boxed{X \sim \text{Pareto}\left(\alpha^* = 1,\ \theta^* = \frac{1}{\theta}\right)}$$

In this problem:

$$\mathbb{P}[X \leq 3] = 1 - \frac{\frac{1}{5}}{3+\frac{1}{5}} = 1 - \frac{\frac{1}{\cancel{5}}}{\frac{16}{\cancel{5}}} = \frac{15}{16} \quad \blacksquare$$

Example. [A GENERALIZATION OF THE ABOVE CASE]

$$\begin{cases} X \mid \Lambda = \lambda \sim \text{Exp}\left(\text{mean} = \frac{1}{\lambda}\right) \\ \Lambda \sim \text{Gamma}(\alpha, \theta) \end{cases}$$

$$\Rightarrow X \sim \text{Pareto}\left(\alpha^* = \alpha,\ \theta^* = \frac{1}{\theta}\right)$$

For the proof, see Example 5.4. in the book.

Example.

$$\begin{cases} X \mid \Theta = \theta \sim \text{Normal}(\text{mean} = \theta, \text{var} = \nu^2) \\ \Theta \sim \text{Normal}(\text{mean} = \mu, \text{var} = \sigma^2) \end{cases}$$

$$X \sim \text{Normal}(\text{mean} = \underline{\mu}, \text{var} = \underline{\sigma^2 + \nu^2})$$

w/ $\cdot\ \mathbb{E}[X] = \mathbb{E}[\underbrace{\mathbb{E}[X \mid \Theta]}_{= \Theta}] = \mathbb{E}[\Theta] = \mu$

$$\cdot\ \text{Var}[X] = \text{Var}[\mathbb{E}[X \mid \Theta]] + \mathbb{E}[\text{Var}[X \mid \Theta]]$$

$$= \text{Var}[\Theta] + \nu^2$$

$$= \sigma^2 + \nu^2$$

*211. An actuary for a medical device manufacturer initially models the failure time for a particular device with an exponential distribution with mean 4 years.

This distribution is replaced with a spliced model whose density function:

(i) is uniform over [0, 3]

(ii) is proportional to the initial modeled density function after 3 years

(iii) is continuous

Calculate the probability of failure in the first 3 years under the revised distribution.

(A) 0.43

(B) 0.45

(C) 0.47

(D) 0.49

(E) 0.51

212. For an insurance:

(i) The number of losses per year has a Poisson distribution with $\lambda = 10$.

(ii) Loss amounts are uniformly distributed on (0, 10).

(iii) Loss amounts and the number of losses are mutually independent.

(iv) There is an ordinary deductible of 4 per loss.

Calculate the variance of aggregate payments in a year.

(A) 36

(B) 48

(C) 72

(D) 96

(E) 120

6.

→: The old model has the density f'tion

$$f_T(x) = \frac{1}{4} e^{-\frac{x}{4}} \quad , x > 0$$

The new model has the density f'tion

$$\tilde{f}_T(x) = \begin{cases} c & \text{for } 0 \le x \le 3 \\ \kappa \cdot e^{-\frac{x}{4}} & \text{for } x > 3 \end{cases}$$

w/ c and κ chosen so that

- $\tilde{f}_T$ is continuous @ 3
- $\tilde{f}_T$ integrates to 1.

$$c = \kappa \cdot e^{-\frac{3}{4}}$$

$$1 = \int_0^{+\infty} \tilde{f}_T(x)\,dx = \int_0^3 c\,dx + \int_3^{+\infty} \kappa e^{-\frac{x}{4}} dx$$

$$1 = 3c + \kappa(-4) e^{-\frac{x}{4}} \Big|_{x=3}^{+\infty}$$

$$1 = 3c + 4\kappa(0 + e^{-3/4}) = 3c + 4\kappa e^{-3/4}$$

$$1 = 3c + 4c \quad \Rightarrow \quad \boxed{c = \frac{1}{7}}$$

⇒ The probab. we need is

$$3\left(\frac{1}{7}\right) = \frac{3}{7} = 0.43 \Rightarrow (A)$$