Name:

M339J Probability Models with Actuarial Applications
University of Texas at Austin
In-Term Exam I
Instructor: Milica Čudina

Notes: This is a closed book and closed notes exam. The maximal score on this exam is 85 points.
Time: 50 minutes

TRUE/FALSE

1.3 (2)	TRUE	FALSE
1.4 (2)	TRUE	FALSE
1.5 (2)	TRUE	FALSE
1.6 (2)	TRUE	FALSE
1.7 (2)	TRUE	FALSE

MULTIPLE CHOICE

1.11 (5)	a	b	c	d	e
1.12 (5)	a	b	c	d	e
1.13 (5)	a	b	c	d	e
1.14 (5)	a	b	c	d	e
1.15 (5)	a	b	c	d	e

FOR GRADER'S USE ONLY:

DEF'N	T/F	1.8	1.9	1.10	M.C.	Σ

1.1. DEFINITIONS.

Problem 1.1. (10 points) Provide the definition of the *cumulative distribution function* of a random variable X.

Solution: The *cumulative distribution function* of a random variable X is the function $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \mathbb{P}[X \leq x] \quad \text{for every } x \in \mathbb{R}. \tag{1.1}$$

Problem 1.2. (10 points) Provide the definition of a *continuous random variable*.

Solution: A random variable X is said to be *continuous* if its cumulative distribution function $F_X : \mathbb{R} \to [0,1]$ satisfies the following properties:

- it is continuous everywhere; and
- it is differentiable everywhere with the possible exceptions of at most countably many values.

1.2. **TRUE/FALSE QUESTIONS.**

Please, note your chosen answer on the front page of this exam!

Problem 1.3. (2 pts) Let X have a Pareto distribution with parameters α and θ. Then, $\mathbb{E}[X^k]$ exists for every $k = 1, 2, \dots$. *True or false?*

Solution: FALSE

Problem 1.4. (2 pts) Let the random variable X represent the outcome of rolling a fair dodecahedron (i.e., a 12 sided "die") with faces numbered 1 through 12. Then the expecation of X equals 6. *True or false?*

Solution: FALSE

$$\mathbb{E}[X] = \frac{1}{12}[1 + 2 + \cdots + 12] = \frac{1}{12} \cdot \frac{12 \cdot 13}{2} = \frac{13}{2}\,.$$

Problem 1.5. (2 points) Let N be an $\mathbb{N}_0$−valued random variable with the probability generating function P_N. As usual, p_N denotes the probability mass function of the random variable N. Then,

$$p_N(0) = P_N(0).$$

True or false?

Solution: TRUE
In fact, $P_N(0) = \mathbb{E}[0^N] = 0^0 p_N(0) = p_N(0)$.

Problem 1.6. (2 points) For a random variable X and for a positive constant d, in our usual notation, we have

$$\mathbb{E}[X] = e_X(d)S_X(d) + \mathbb{E}[X \wedge d]\,. \tag{1.2}$$

True or false?

Solution: TRUE

Problem 1.7. (2 pts) Let X have the loglogistic distribution. Then, the random variable $X' = 1/X$ also has the loglogistic distribution. *True or false?*

Solution: TRUE
From the tables, the cdf of X can be written as

$$F_X(x) = \frac{(x/\theta)^\gamma}{1 + (x/\theta)^\gamma}, \qquad x > 0,$$

for parameters γ and θ.

In class, we learned that for $y > 0$,

$$\begin{aligned} F_{X'}(y) &= 1 - F_X(1/y) \\ &= 1 - \frac{(1/y\theta)^\gamma}{1 + (1/y\theta)^\gamma} \\ &= \frac{1}{1 + (1/y\theta)^\gamma} \\ &= \frac{(y/\theta^*)^\gamma}{1 + (y/\theta^*)^\gamma} \end{aligned}$$

with $\theta^* = 1/\theta$.

1.3. **Free-response problems.** Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they're correct) are worth 0 points.

Problem 1.8. (10 points) A population of insureds consists of three types of people: α, β and γ. There are half as many people of Type α as of Type β people in the population. The number of Type γ people is equal to the total number of the remaining two types of people. The probability that a Type α person makes at least one claim in a year is $1/5$. The probability that a Type β person makes at least one claim in a year is $2/5$. The probability that a Type γ person makes at least one claim in a year is $3/5$.

At the end of the year, a person is chosen at random from the population. It is observed that that person had at least one claim. What is the probability that the person was of Type β?

Solution: From the given breakdown of the population, we conclude that

$$\mathbb{P}[\alpha] = 1/6, \quad \mathbb{P}[\beta] = 1/3, \quad \mathbb{P}[\gamma] = 1/2. \tag{1.3}$$

Let E denote the event that there was at least one claim. By Bayes' Theorem, we have that

$$\begin{aligned}\mathbb{P}[\beta \,|\, E] &= \frac{\mathbb{P}[E \,|\, \beta] \times \mathbb{P}[\beta]}{\mathbb{P}[E \,|\, \alpha] \times \mathbb{P}[\alpha] + \mathbb{P}[E \,|\, \beta] \times \mathbb{P}[\beta] + \mathbb{P}[E \,|\, \gamma] \times \mathbb{P}[\gamma]} \\ &= \frac{(2/5)(1/3)}{(1/5)(1/6) + (2/5)(1/3) + (3/5)(1/2)} = \frac{2}{7}\,.\end{aligned} \tag{1.4}$$

Problem 1.9. (5 points) Assume that you are building a model for the ground-up loss random variable for home insurance. The homes you are insuring fall into two categories: mansions and cottages. Propose **one distribution** as a model for the ground-up loss you will be using for every home in your entire "population" of insured homes. Explain your reasoning in one or two sentences.

Solution: Any $2-$point mixture of distributions with positive supports would work.

Problem 1.10. (15 points) Let X have a Pareto distribution with parameters α and θ.
Define $Y = \ln(1 + \frac{X}{\theta})$.
Describe the distribution of the random variable Y, i.e., give the **name** of the distribution and the **values** of its parameters.

Solution: The distribution function of Y is obtained as follows: for every $y > 0$,

$$\begin{aligned} F_Y(y) = \mathbb{P}[Y \leq y] &= \mathbb{P}[\ln(1 + \frac{X}{\theta}) \leq y] \\ &= \mathbb{P}[1 + \frac{X}{\theta} \leq e^y] \\ &= \mathbb{P}[\frac{X}{\theta} \leq e^y - 1] \\ &= \mathbb{P}[X \leq \theta(e^y - 1)]. \end{aligned}$$

Using the given distribution of X, we have

$$F_Y(y) = F_X(\theta(e^y - 1)) = 1 - (\frac{\theta}{\theta(e^y - 1) + \theta})^\alpha = 1 - e^{-\alpha y}.$$

We conclude that Y is exponential with the parameter $\theta^* = 1/\alpha$.

1.4. MULTIPLE CHOICE QUESTIONS.

Please, note your chosen answer on the front page of this exam!

Problem 1.11. (5 pts) Let the severity random variable X be continuous such that $f_X(x) > 0$ for all $x > 0$. Let Y^L denote the **per loss** random variable associated with X for some ordinary deductible d. Then the random variable Y^L is

(a) continuous.
(b) discrete, but not degenerate (constant).
(c) degenerate (constant).
(d) mixed.
(e) None of the above

Solution: (d)
The c.d.f. of Y^L has a single jump at 0.

Problem 1.12. (5 pts) *Source: Prof. Jim Daniel (personal communication).*
The ground-up loss X is modeled by an exponential distribution with mean \$500. There is an ordinary deductible of $d = 100$. What can you say about the expected value of the per-loss random variable?

(a) It is less than 100.
(b) It is more than 100, but less than 250.
(c) It is more than 250, but less than 375.
(d) It is more than 375, but less than 500.
(e) None of the above

Solution: (d)
Let

$$Y^L = (X - d)_+$$

with $X \sim Exp(\theta = 500)$ and $d = 100$. Then,

$$\begin{aligned}
\mathbb{E}[Y^L] &= \mathbb{E}[(X-d)\mathbb{I}_{[X>d]}] \\
&= \int_d^\infty (x-d)\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx \\
&= \int_0^\infty y\frac{1}{\theta}e^{-\frac{y+d}{\theta}}\,dx \\
&= e^{-\frac{d}{\theta}}\int_0^\infty y\frac{1}{\theta}e^{-\frac{y}{\theta}}\,dx \\
&= \theta e^{-\frac{d}{\theta}} = 500e^{-1/5} \approx 409.37.
\end{aligned}$$

Problem 1.13. (5 pts) Given that the hazard rate function of a random variable X equals $h_X(x) = 0.3x$ for $x > 0$, we can conclude that

(a) $0 \leq f_X(2) < 1/4$
(b) $1/4 \leq f_X(2) < 1/3$
(c) $1/3 \leq f_X(2) < 1/2$
(d) $1/2 \leq f_X(2) < 1$
(e) None of the above.

Solution: (b)
The survival function of X is

$$\begin{aligned} S_X(x) &= e^{-\int_0^x h_X(z)\,dz} \\ &= e^{-0.3\int_0^x z\,dz} \\ &= e^{-0.3\cdot\frac{1}{2}z^2|_{z=0}^{x}} \\ &= e^{-0.15x^2}, \qquad x \geq 0. \end{aligned}$$

So, the density of X is given by

$$f_X(x) = -S'_X(x) = -(-0.15)\cdot 2x \cdot e^{-0.15x^2} = 0.3\,x\,e^{-0.15x^2}, \qquad x \geq 0.$$

In particular,

$$f_X(2) = 0.6e^{-0.6} \approx 0.3293.$$

Problem 1.14. (5 points) A continuous random variable X has the *probability density function* of the following form

$$f_X(x) = \kappa x^{-5} \quad \text{for } x > 1$$

where κ is a positive constant. Let x_* denote its 95^{th} percentile. Then,

(a) $0 \leq x_* < 1$
(b) $1 \leq x_* < 2$
(c) $2 \leq x_* < 3$
(d) $3 \leq x_* < 4$
(e) None of the above.

Solution: (c)

$$x^* = 2.115 \tag{1.5}$$

Problem 1.15. (5 pts) Let the severity random variable X be continuous such that $f_X(x) > 0$ for all $x > 0$. Let Y^P denote the per payment random variable associated with X for some ordinary deductible $d > 0$. Then the random variable Y^P is

(a) continuous.
(b) discrete, but not degenerate (constant).
(c) degenerate (constant).
(d) mixed.
(e) None of the above

Solution: (a)