Subjective Probabilities

(W) | August 31st, 2018.

Individual investor **forms** an **opinion** about the probability distribution of the time-T stock price $\boxed{S(T)}$.

At the **very least**:

$S(T)$ ... random variable denotes the asset price @ time T

For now, focus on the investor's beliefs w/ respect to $\mathbb{E}[S(T)]$.

Assume: Invest in a portfolio (among the admissible ones) which has the highest Expected Profit.

UNIVERSITY OF TEXAS AT AUSTIN

Subjective probabilities.

Problem 1.1. IFM Sample (Introductory) Problem #6.
The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- An investor who decides to long the forward contract denotes by P the expected stock price in one year.

Determine which of the following statements about P is **TRUE**.

(A) $P < 100$
(B) $P = 100$
(C) $100 < P < 105$
(D) $P = 105$
(E) $P > 105$

$$\mathbb{E}[\underbrace{\text{Profit}}] > 0$$

$$\text{Profit} = \text{Payoff} = S(T) - F$$

$$\Rightarrow \mathbb{E}[S(T)] - F > 0$$

$$\Rightarrow P > F = 105$$

Problem 1.2. IFM Sample (Introductory) Problem #38. $\mathbb{E}[S(3)] = 90$

The current price of a medical company's stock is 75. The expected value of the stock price in three years is 90 per share. The stock pays no dividends. You are also given:

- The risk-free interest rate is positive. $r > 0$
- There are no transaction costs.
- Investors require compensation for risk. $\Rightarrow \mathbb{E}[\text{Profit}] > 0$

The price of a three-year forward on a share of this stock is X, and at this price an investor is willing to enter into the forward. Determine what can be concluded about X.

(A) $X < 75$
(B) $X = 75$
(C) $75 < X < 90$
(D) $X = 90$
(E) $X > 90$

$90 = \mathbb{E}[S(3)] >$ forward price $= X$ (as above!)

- $F_{0,3}(S) = \underbrace{S(0)e^{(r-\delta)\cdot 3}}_{\text{always true for forwards on stocks}} = S(0)e^{3r} > S(0) = 75$ in this problem $r>0$, $\delta = 0$

2.

Problem 1.3. IFM Sample (Introductory) Problem #70.
Investors in a certain stock demand to be compensated for risk. The current stock price is 100. The stock pays dividends at a rate proportional to its price. The dividend yield is 2%. The continuously compounded risk-free interest rate is 5%. Assume there are no transaction costs.

Let X represent the expected value of the stock price 2 years from today. Assume it is known that X is a whole number. Determine which of the following statements is true about X.

(A) The only possible value of X is 105.
(B) The largest possible value of X is 106.
(C) The smallest possible value of X is 107.
(D) The largest possible value of X is 110.
(E) The smallest possible value of X is 111.

$\delta = 0.02$

$r = 0.05$

· Investor invests in 1 share: Initial cost is $S(0)$

· At time 2: The investor owns $e^{0.02 \cdot 2}$ shares:

$$\mathbb{E}[\text{Profit}] = \mathbb{E}[e^{\delta \cdot T} \cdot S(T) - S(0)e^{rT}] > 0$$

$$\Rightarrow \quad e^{0.04} \cdot \mathbb{E}[S(T)] > 100e^{0.05 \cdot 2}$$

$$\Rightarrow \quad X := \mathbb{E}[S(2)] > 100e^{0.06} = 106.18.$$

$\Rightarrow$ (C).

3.

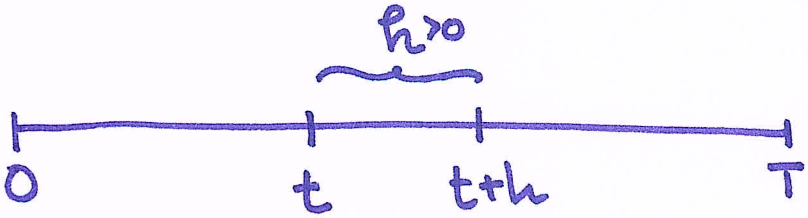
Recall:

for n periods: the length of every period $h_n = \frac{T}{n}$

→ returns are independent between periods

→ returns are identically distributed for different periods (which are, by design, of equal length).

$\boxed{\text{Now:}}$



For every t, h:

define the realized return (a random variable)

$$R(t, t+h) := \ln\left(\frac{S(t+h)}{S(t)}\right)$$

Recall the growth of money under r... ccrfir:

$$a(t)e^{r \cdot h} = a(t+h)$$

accumulation f'tion

Note: $S(t+h) = S(t)e^{R(t,t+h)}$

We require:

- for $\underbrace{(t, t+h) \text{ and } (t+h, t+h+\varepsilon)}_{\text{disjoint time intervals}}$:

  $R(t, t+h)$ and $R(t+h, t+h+\varepsilon)$ Independent

- for $(t, t+h)$ and $(s, s+h)$

  $R(t, t+h)$ and $R(s, s+h)$ are identically distributed

We have:

- for $(t, t+s)$ and $(t+s, t+s+h)$

$$R(t, t+s) + R(t+s, t+s+h) = \ln\left(\frac{S(t+s)}{S(t)}\right) + \ln\left(\frac{S(t+s+h)}{S(t+s)}\right)$$

$$= \ln\left(\frac{\cancel{S(t+s)}}{S(t)} \cdot \frac{S(t+s+h)}{\cancel{S(t+s)}}\right) = \ln\left(\frac{S(t+s+h)}{S(t)}\right)$$

$$= R(t, t+s+h)$$

We say that realized returns are ADDITIVE!

We will model the realized returns using the normal dist'n; all we need to look @:

$$R(0, t) \sim \text{Normal}(\text{mean} = \mu, \text{var} = \sigma^2)$$

We already have:

σ... the volatility parameter !

We immediately know:

$$Var[R(0,1)] = \sigma^2, \quad \text{i.e.,} \quad \sigma = SD[R(0,1)]$$

Q: What is the volatility over a period of length h?

$$\boxed{\sigma\sqrt{h}}$$

$$\Rightarrow \quad Var[R(0,t)] = \tau^2 = \sigma^2 \cdot t$$