W: April 5th, 2019.

9. Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The current stock price is 50.

(iii) The current call option delta is 0.61791.

(iv) There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

$\Downarrow$

$ds = S(0) \cdot \sigma \sqrt{h}$

w/ $h = \frac{1}{365}$

(A) 0.41

(B) 0.52

(C) 0.63

(D) 0.75

(E) 1.11

10-17. DELETED

18. A market-maker sells 1,000 1-year European gap call options, and delta-hedges the position with shares.

You are given:

(i) Each gap call option is written on 1 share of a nondividend-paying stock.

(ii) The current price of the stock is 100.

(iii) The stock's volatility is 100%.

(iv) Each gap call option has a strike price of 130.

(v) Each gap call option has a payment trigger of 100.

(vi) The risk-free interest rate is 0%.

Under the Black-Scholes framework, determine the initial number of shares in the delta-hedge.

1.

Given that

$$\Delta_c(S(0),0) = 0.61791 \cong 0.6179.$$

In general,

$$\Delta_c(S(0),0) = \underbrace{e^{-\delta\cdot T}}_{=1} \cdot N(d_1(S(0),0)) = 0.6179$$

no dividends

tables

$$\Rightarrow \quad d_1(S(0),0) = N^{-1}(0.6179) = 0.3$$

by def'n → ||

at the money

$$\frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{S(0)}{K}\right)} + \left(r+\frac{\sigma^2}{2}\right)\cdot T\right]$$

$$\frac{1}{\sigma\sqrt{1/4}}\left[0.10+\frac{\sigma^2}{2}\right]\cdot\frac{1}{4} = 0.3 \qquad /\cdot\sigma$$

$$\left(0.10+\frac{\sigma^2}{2}\right)\cdot\sqrt{1/4} = 0.3\cdot\sigma$$

$$\frac{\sigma^2}{2}+0.10 = 0.6\cdot\sigma \qquad /\cdot 2$$

$$\sigma^2 - 1.2\sigma + 0.2 = 0$$

One of the solutions is, immediately, $\sigma_1 = 1$

$$\Rightarrow \quad (\sigma-1)(\sigma-0.2) = 0$$

Discard $\sigma=1$ as a sol'n; just keep $\sigma=0.2$

$$\Rightarrow \text{answer}: \quad 50\cdot 0.2\cdot\sqrt{\frac{1}{365}} = 10\cdot\frac{1}{\sqrt{365}} \cong 0.52$$

2.

Delta-Gamma Hedging:

Let your investor start w/ a delta-neutral portfolio. So, w/ the value function of the investor's portfolio denoted by $v(s,t)$, (s)he **maintains**

$$\boxed{\Delta(s,t) = \frac{\partial}{\partial s} v(s,t) = 0}$$

This can be accomplished by trading in the shares of the underlying continuously.

Then, the investor decides to Γ-hedge as well, i.e., (s)he wants to create a **Γ-neutral** portfolio.

Q: Can they accomplish this by simply trading in the shares of stock?

→: **Remember**: the Γ of the stock is 0.

So, another option w/ non-zero Γ is needed.

Let the price of such an option be denoted by $\tilde{v}(s,t)$. **Then**:

- $\tilde{\Delta}(s,t) = \frac{\partial}{\partial s}\tilde{v}(s,t)$
- $\tilde{\Gamma}(s,t) = \frac{\partial^2}{\partial s^2}\tilde{v}(s,t)$.

Let $\tilde{n}(s,t)$ be the number of these options to hold @ time-t w/ the stock price s. Then, to have Γ-neutrality, it must satisfy:

$$\Gamma(s,t) + \tilde{n}(s,t)\cdot\tilde{\Gamma}(s,t) = 0$$

w/ $\Gamma(s,t) = \frac{\partial^2}{\partial s^2} v(s,t)$

$$\Rightarrow \quad \tilde{n}(s,t) = -\frac{\Gamma(s,t)}{\tilde{\Gamma}(s,t)}$$

Now: the Δ of the total portfolio will be

$$\Delta(s,t) + \tilde{n}(s,t)\cdot\tilde{\Delta}(s,t)$$

To re-establish Δ-neutrality, we need to trade in shares of stock. Let $n_S(s,t)$ be the required # of shares. It must satisfy:

$$\underbrace{\Delta(s,t)}_{=0} + \tilde{n}(s,t)\cdot\tilde{\Delta}(s,t) + n_S(s,t) = 0$$

the original portfolio was Δ-neutral

Δ-neutrality

$$\Rightarrow \quad n_S(s,t) = -\tilde{n}(s,t)\cdot\tilde{\Delta}(s,t)$$

Again, the Γ of the stock investment is $\underline{\underline{0}}$. So, Γ-neutrality is maintained. ∎

10. For two European call options, Call-I and Call-II, on a stock, you are given:

n_I n_{II} n_S

Greek	Call-I	Call-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746
Vega	0.0781	0.0596

Suppose you just sold 1000 units of Call-I. $\boxed{n_I = -1000}$

Determine the numbers of units of Call-II and stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in Call-I.

(A) buy 95.8 units of stock and sell 872.7 units of Call-II ✗

(B) sell 95.8 units of stock and buy 872.7 units of Call-II

(C) buy 793.1 units of stock and sell 692.2 units of Call-II ✗

(D) sell 793.1 units of stock and buy 692.2 units of Call-II ✗

(E) sell 11.2 units of stock and buy 763.9 units of Call-II ✗

Δ-neutrality: $(-1000)\cdot 0.5825 + n_{II}\cdot(0.7773) + n_S = 0$

Γ-neutrality: $(-1000)(0.0651) + n_{II}(0.0746) = 0$

$$\Rightarrow n_{II} = \frac{651}{0.0746} = 872.7 \text{ (long)}$$

$\Rightarrow$ (B) ■

Verify by calculating n_S

$n_S = 582.5 - 872.7 \cdot 0.7773 = -95.8$ ✓ or ■

5.

Exchange Options.

$T \ldots$ exercise date

two risky assets: $\begin{cases} S \ldots \text{underlying asset} \\ Q \ldots \text{strike asset} \end{cases}$

For an exchange call:

the payoff: $V_{EC}(T,S,Q) = (S(T) - Q(T))_+$

For an exchange put:

$$V_{EP}(T,S,Q) = (Q(T) - S(T))_+$$

We have a special symmetry:

$$\boxed{V_{EC}(T,S,Q) = V_{EP}(T,Q,S)}$$

$\Rightarrow$ The prices at time 0 must be equal, i.e.,

$$\boxed{V_{EC}(0,S,Q) = V_{EP}(0,Q,S)}$$

$\Rightarrow$ It's sufficient to develop the Black-Scholes pricing formula for EXCHANGE CALLS.

- S ... underlying ; has δ_S, σ_S

Because we're pricing, we look at the its time-T price under the <u>risk-neutral measure</u>:

In the Black-Scholes model:

$$S(T) = S(0)\, e^{\left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot T + \sigma_S \sqrt{T} \cdot \tilde{Z}_S}$$

- Q ... strike asset; has δ_Q, σ_Q

$$Q(T) = Q(0)\, e^{\left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot T + \sigma_Q \cdot \sqrt{T} \cdot \tilde{Z}_Q}$$

w/ $\tilde{Z}_S$ and $\tilde{Z}_Q$ std normal rnd variables

w/ ρ ... the correlation coefficient between Z_S and Z_Q