Exchange Options (cont'd).

W: April 8^{th}, 2019.

S… underlying δ_S, σ_S, driving Z_S

Q… strike asset δ_Q, σ_Q, driving Z_Q

$\rho = corr[Z_S, Z_Q]$

Black·Scholes pricing formula (master).

$$V_{EC}(0) = F^P_{0,T}(S)\cdot N(d_1) - F^P_{0,T}(Q)\cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(Q)}\right) + \frac{1}{2}\sigma^2\cdot T\right]$$

and $d_2 = d_1 - \sigma\sqrt{T}$.

where $\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\cdot\sigma_S\cdot\sigma_Q$

Note:
- We can price exchange puts by changing the roles of S and Q.
- This pricing formula simplifies to the vanilla B·S price when Q is a riskless asset.
- $S(t), t \geq 0$

  $Q(t), t \geq 0$

  For every t:

$$\left(Var\left[\ln\left(\frac{S(t)}{Q(t)}\right)\right]=\right) Var\left[\ln(S(t)) - \ln(Q(t))\right] = \text{(under } \mathbb{P}^*\text{)}$$

deterministic

$$= Var\Big[\ln(S(0)) + \left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot t + \sigma_S\cdot\sqrt{t}\cdot Z_S$$
$$- \ln(Q(0)) - \left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot t - \sigma_Q\sqrt{t}\cdot Z_Q\Big]$$

1.

$$= Var\left[\sigma_S \underbrace{\sqrt{t}} \cdot Z_S - \sigma_Q \cdot \underbrace{\sqrt{t}} \cdot Z_Q\right] =$$

$$= t \cdot Var\left[\sigma_S Z_S - \sigma_Q Z_Q\right] =$$

$$= t \cdot \Big(\sigma_S^2 \cdot \underbrace{Var[Z_S]}_{=1} + \sigma_Q^2 \cdot \underbrace{Var[Z_Q]}_{=1}$$

$$- 2\sigma_S \cdot \sigma_Q \cdot \underbrace{Cov[Z_S, Z_Q]}_{= \rho \cdot SD[Z_S] \cdot SD[Z_Q] = \rho} \Big)$$

$$= t\underbrace{(\sigma_S^2 + \sigma_Q^2 - 2\sigma_S \cdot \sigma_Q \cdot \rho)} = t \cdot \sigma^2$$

- Continuous dividend paying stocks S and Q:

$$V_{EC}(0) = S(0)e^{-\delta_S \cdot T} \cdot N(d_1) - Q(0)e^{-\delta_Q \cdot T} \cdot N(d_2)$$

w/ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)e^{-\delta_S \cdot T}}{Q(0)e^{-\delta_Q \cdot T}}\right) + \frac{1}{2}\sigma^2 \cdot T\right]$

$$\Rightarrow d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{Q(0)}\right) + \left(\underbrace{\delta_Q}_{r \dots ccrfir} - \underbrace{\delta_S}_{\delta \dots \text{underlying asset}} + \frac{\sigma^2}{2}\right) \cdot T\right]$$

and $d_2 = d_1 - \sigma\sqrt{T}$

where $\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\sigma_S \cdot \sigma_Q \cdot \rho$

54. Assume the Black-Scholes framework. Consider two nondividend-paying stocks whose time-t prices are denoted by $S_1(t)$ and $S_2(t)$, respectively.

You are given:

(i) $S_1(0) = 10$ and $S_2(0) = 20$.

(ii) Stock 1's volatility is 0.18. $\Rightarrow \sigma_1 = 0.18$

(iii) Stock 2's volatility is 0.25. $\Rightarrow \sigma_2 = 0.25$

(iv) The correlation between the continuously compounded returns of the two stocks is –0.40. $\Rightarrow \rho = -0.40$

(v) The continuously compounded risk-free interest rate is 5%. $r = 0.05$

(vi) A one-year European option with payoff $\max\{\min[2S_1(1), S_2(1)] - 17, 0\}$ has a current (time-0) price of 1.632.

SPECIAL OPTION: "PUT"

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now.

$T=1$

Calculate the current (time-0) price of this option.

(A) 0.67

(B) 1.12

(C) 1.49

(D) 5.18

(E) 7.86

The payoff of the special option ("put"):

$$\Big(17 - \underbrace{\min(2S_1(1), S_2(1))}_{=Y(1)}\Big)_+ \quad (SP)$$

$\Rightarrow$ This is, indeed, a put option on Y w/ strike 17.

(vi) gives us the price of the corresponding call option on Y. (SC)

$\Rightarrow$ We use put-call parity:

$$\underset{\substack{\| (vi) \\ 1.632}}{V_{SC}(0)} - \underset{\substack{\updownarrow \\ ???}}{V_{SP}(0)} = \boxed{\underset{???}{F^P_{0,T}(Y)}} - \underset{\substack{\| \\ 17e^{-0.05}}}{PV_{0,T}(K)} \qquad 3.$$

Focus on $F^P_{0,T}(Y)$: this is the price we have to pay @ time·0 to receive

$Y(1) = \min(2\cdot S_1(1), S_2(1))$ @ time·1.

$\boxed{\min(a,b) = -\max(-a,-b)}$

$$Y(1) = \min(2S_1(1), S_2(1))$$

$$= S_2(1) + \min(2S_1(1) - S_2(1), 0)$$

$$= \underbrace{S_2(1)}_{\substack{\uparrow \\ \text{prepaid} \\ \text{forward}}} - \underbrace{\max(S_2(1) - 2S_1(1), 0)}_{\substack{\uparrow \\ \text{exchange call} \\ \text{w/ underlying } S_2 \\ \text{and strike asset } 2S_1}}$$

$$2\cdot S_1(T) = 2\cdot S_1(0)\, e^{(r-\delta_1-\frac{\sigma_1^2}{2})\cdot T + \sigma_1\sqrt{T}\cdot Z_1}$$

It's the same δ_1 and σ_1 as for the original stock.

At time·0 :
- the prepaid forward is priced @ $S_2(0)$ (no dividends)
- we use the B·S formula to price the exchange call