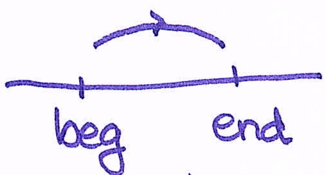
Section 11.1. The expected return of a portfolio

i … $i = 1..n$; indices of the investment components of a portfolio

For every i: R_i … the realized return of the i^{th} component over our period (say, a year)

Let R_p .. the realized return of the entire portfolio.



$$R_p := \frac{P_p^{end} - P_p^{beg}}{P_p^{beg}} = ?$$

P_p … price of the portfolio

$$P_p = \sum_{i=1}^{n} \underbrace{P_i}$$

↑ the values of the component i in the portfolio

$$R_p = \frac{\sum_{i=1}^{n} P_i^{end} - \sum_{i=1}^{n} P_i^{beg}}{P_p^{beg}} = \sum_{i=1}^{n} \frac{P_i^{end} - P_i^{beg}}{P_p^{beg}} \cdot \frac{P_i^{beg}}{P_i^{beg}}$$

$$R_p = \sum_{i=1}^{n} \underbrace{\frac{P_i^{end} - P_i^{beg}}{P_i^{beg}}}_{= R_i} \cdot \frac{P_i^{beg}}{P_p^{beg}} =: x_i$$

x_i … portfolio weight of investment i

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of the portfolio}} \Rightarrow \boxed{\sum_{i=1}^{n} x_i = 1}$$

$$\Rightarrow \boxed{R_p = x_1 \cdot R_1 + \dots + x_n \cdot R_n = \sum_{i=1}^{n} x_i \cdot R_i}$$

$\Rightarrow$ The expected return of the portfolio:

$$\mathbb{E}[R_p] = \sum_{i=1}^{n} x_i \cdot \mathbb{E}[R_i]$$

4) You are given the following information about a portfolio consisting of stocks X, Y, and Z:

Stock	Investment	Expected Return
X	10,000	8% ✓
Y	15,000	12% ✓
Z	25,000	16% ✓

$\Sigma = 50{,}000$

Calculate the expected return of the portfolio.

(A) 10.8%

(B) 11.4%

(C) 12.0%

(D) 12.6%

(E) 13.2%

$$x_X = \frac{10{,}000}{50{,}000} = 0.2$$

$$x_Y = \frac{15{,}000}{50{,}000} = 0.3$$

$$x_Z = \frac{25{,}000}{50{,}000} = 0.5$$

$\Rightarrow$ answer:

$$0.2 \cdot 0.08 + 0.3 \cdot 0.12 + 0.5 \cdot 0.16 = 0.132$$

11.2. The volatility of a Two-Stock Portfolio

$\boxed{\text{Covariance}}$ $i \neq j$

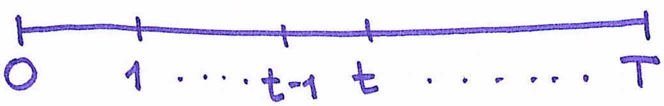
Look @ $\underline{R_i \text{ and } R_j}$ - returns of components i & j.

Then,

$$\left.\begin{aligned} Cov[R_i, R_j] &= \mathbb{E}\left[(R_i - \mathbb{E}[R_i])(R_j - \mathbb{E}[R_j])\right] \\ &= \mathbb{E}[R_i \cdot R_j] - \mathbb{E}[R_i] \cdot \mathbb{E}[R_j] \end{aligned}\right\}$$

Usually, we do not have theoretical values of our parameters; including the covariance.

So, we create estimators:



Set for every t: Return over $(t-1, t)$ is $R_{i,t}$ for i and $R_{j,t}$ for j.

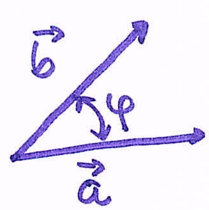
The estimator we use:

$$S^2_{i,j} = \frac{1}{T-1} \sum_{t=1}^{T} (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j)$$

??? Just so!

Correlation:

$$corr[R_{i,j}] = \frac{Cov[R_i, R_j]}{SD[R_i] \cdot SD[R_j]}$$



scalar product:

$$\|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\varphi) = \langle \vec{a}, \vec{b} \rangle$$

Volatility of a Two-Stock Portfolio

$$R_p = x_1 \cdot R_1 + x_2 \cdot R_2$$

$$\Rightarrow Var[R_p] = Var[x_1 \cdot R_1 + x_2 \cdot R_2]$$

$$= x_1^2 \cdot Var[R_1] + \underline{\underline{2}}\, x_1 \cdot x_2 \cdot Cov[R_1, R_2] + x_2^2 \cdot Var[R_2]$$

$\Rightarrow$ The volatility of the portfolio:

$$\boxed{SD[R_p] = \sqrt{Var[R_p]}}$$

3) You are given the following information about the annual returns of two stocks, X and Y:

i) The expected returns of X and Y are $\overset{r_x}{\cancel{\bar{R}_X}} = 10\%$ and $\overset{r_y}{\cancel{\bar{R}_Y}} = 15\%$.

ii) The volatilities of the returns are $\overset{\sigma_X}{\cancel{\sigma_X}} = 18\%$ and $\overset{\sigma_Y}{\cancel{\sigma_Y}} = 20\%$.

iii) The correlation coefficient of the returns for these two stocks is 0.25. $= \rho_{X,Y}$

iv) The expected return for a certain portfolio, consisting only of stocks X and Y, is 12%.

w_X... weight of stock X investment
w_Y... weight of stock Y investment

Calculate the volatility of the portfolio return.

(A) 10.88%

(B) 12.56%

(C) 13.55%

(D) 14.96%

(E) 16.91%

(iv) ⇒

$$\mathbb{E}[R_P] = w_X \cdot \mathbb{E}[R_X] + w_Y \, \mathbb{E}[R_Y]$$

$$= w_X \cdot 0.10 + (1-w_X) \cdot 0.15 = 0.12$$

$$w_X(0.10 - 0.15) = 0.12 - 0.15)$$

$$\boxed{w_X = \frac{0.03}{0.05} = 0.6} \Rightarrow w_Y = 0.4$$

$$Var[R_P] = (0.6)^2 \cdot (0.18)^2 + 2 \cdot 0.6 \cdot 0.4 \cdot 0.25 \cdot 0.18 \cdot 0.2 + (0.4)^2 \cdot (0.2)^2$$

$$= \ldots = 0.02238$$

$$\Rightarrow \boxed{\sigma_P = 0.1496}$$ ∎

11.3. The Volatility of a Large Portfolio

$$R_p = \underbrace{x_1 \cdot R_1 + \dots + x_n \cdot R_n}_{} = \sum_{i=1}^{n} x_i \cdot R_i$$

Then,

$$\begin{aligned} Var[R_p] &= Cov[R_p, R_p] \\ &= Cov[x_1 \cdot R_1 + \dots + x_n \cdot R_n, R_p] \\ &= \sum_{i=1}^{n} x_i \cdot Cov[R_i, R_p] \end{aligned}$$

$\Rightarrow$ We can interpret the variance of the portfolio's return as a <u>weighted</u> average of all the covariances of individual returns w/ the whole portfolio.

2) You are given the following information about a portfolio with four assets.

Asset	Market Value of Asset	Covariance of asset's return with the portfolio return
I	40,000	0.15
II	20,000	-0.10
III	10,000	0.20
IV	30,000	-0.05

Σ 100,000

Calculate the standard deviation of the portfolio return. $\sigma_p = ?$

$$\Rightarrow x_{I} = 0.4;\ x_{II} = 0.2;\ x_{III} = 0.1;\ x_{IV} = 0.3.$$

$$Var[R_p] = \sum_{i=1}^{n} x_i \cdot Cov[R_i, R_p]$$

$$= 0.4 \cdot 0.15 + \cancel{0.2 \cdot (-0.10)} + \cancel{0.1(0.2)} + 0.3(-0.05)$$

$$= 0.045$$

$$\Rightarrow \sigma_p = \sqrt{0.045} \approx 0.212. \Rightarrow (D)$$

(A) 4.50%

(B) 13.2%

(C) 20.0%

(D) 21.2%

(E) 44.7%

We can expand our variance formula as:

$$
\begin{aligned}
Var[R_P] &= Var[x_1 \cdot R_1 + \cdots + x_n \cdot R_n] \\
&= \sum_{i,j} x_i \cdot x_j \cdot Cov[R_i, R_j] \\
&= \sum_{i=1}^{n} x_i^2 \cdot Var[R_i] + \sum_{i \neq j} x_i \cdot x_j \cdot Cov[R_i, R_j] \\
&= \sum_{i=1}^{n} x_i^2 \cdot Var[R_i] + 2 \sum_{i<j} x_i \cdot x_j \cdot Cov[R_i, R_j].
\end{aligned}
$$