Apr 28TH

Understanding the market-maker's

Δ-hedged portfolio

- the SHORT/WRITTEN call option — Just an example, would work for any other option.
- the appropriate number of shares of stock to create a Δ-neutral portfolio

⇒ Have to dynamically rebalance ~~the~~ portfolio!

Assume: The market maker has access to $\Delta_c(t)$ ... the delta of the call as time goes by.

⇒ $\Delta_c(t)$ shares of stock in the Δ-neutral portfolio.

$W(t)$ ... the time-t value of the portfolio

$$\boxed{W(t) = \underset{\text{WRITTEN}}{\underset{\uparrow}{-}} V_c(t) + \Delta_c(t) \cdot S(t)}$$

After a short time dt, the change in the portfolio's worth is:

$$dW(t) = -dV_c(t) + \Delta_c(t)\, dS(t)$$

Remember the Δ-Γ-Θ approximation:

$$dV_c(t) = \Delta_c(t)\, dS(t) + \frac{1}{2}\Gamma_c(t)(dS(t))^2 + \Theta_c(t)\, dt$$

w/ $\Gamma_c(t)$... gamma of the call

$\Theta_c(t)$... theta of the call

SAMPLE MFE

9. Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

$T = \frac{1}{4} \quad K = S(0) = 50 \qquad \delta = 0$

You are given:

(i) The continuously compounded risk-free interest rate is 10%. $r = 0.1$

(ii) The current stock price is 50.

(iii) The current call option delta is 0.61791. $\Delta_C(0) = 0.61791$

(iv) There are 365 days in the year.

← Pretends to be our dt.

If, after one day, the market-maker has zero profit ~~or loss~~, determine the stock price move over the day. ?

$dS(t)$

(A) 0.41

(B) 0.52

(C) 0.63

(D) 0.75

(E) 1.11

$$(dS(t))^2 = \sigma^2 (S(t))^2 dt \qquad /\sqrt{\ }$$

$$\text{ans: } dS(t) = \sigma S(t) \cdot \underbrace{\sqrt{dt}}_{\sqrt{\frac{1}{365}}}$$

$\sigma = ?$

$$\text{Black-Scholes} \Rightarrow \Delta_C(0) = e^{-\delta T} N(d_1) \overset{\delta = 0}{=} N(d_1)$$

$$V_C(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$\text{w/ } d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{S(0)}{K}\right)}^{=0} + \left(r + \frac{1}{2}\sigma^2\right)T\right]$$

$\Rightarrow$ Solve for σ in:

$$\sqrt{\tfrac{1}{4}} = \tfrac{1}{2}$$

$$0.61791 = N\left(\frac{1}{\sigma\cancel{\sqrt{\frac{1}{4}}}}\left(0.1 + \frac{\sigma^2}{2}\right)\cancel{\frac{1}{4}}\right)$$

$$\frac{1}{\sigma}\left(0.1+\frac{\sigma^2}{2}\right)\cdot\frac{1}{2} = 0.3 \quad / \cdot 4\sigma$$

STD NORMAL TABLES.

$$0.2+\sigma^2 = 1.2\sigma$$

$$\sigma^2-1.2\sigma+0.2=0$$

$$(\sigma-1)(\sigma-\text{answer})=0$$

the volatility is $\boxed{\sigma=0.2}$

Final answer: $0.2\cdot 50\cdot\frac{1}{\sqrt{365}} = 0.52 \Rightarrow \boxed{B.}$

Second-order effects

Q: What can you say about Γ for calls and puts in the Black-Scholes model?

$$\Gamma_C = \Gamma_P \underbrace{> 0}$$

$\uparrow$ Put-call parity

$\Rightarrow$ The option price is a CONVEX function of the stock price.

Improvement: If Γ is also used for hedging, i.e., we want to create a portfolio which is BOTH

Δ-neutral $\rightarrow$ O.K. to trade in stocks

and

Γ-neutral $\rightarrow$ We need another option since $\Gamma(\text{stock}) \equiv 0$

The concept:

$W(t)$ ... worth of a delta-neutral portfolio which we want to make also gamma-neutral.

$\Gamma_W(t)$ ... the Γ of the portfolio.

$X(t)$ ... the value of a contingent claim (not currently in our portfolio)

$\Gamma_X(t)$ ... the Γ of the contingent claim w/ $\Gamma_X \neq 0$

$n_X(t)$ ... # of contingent claims needed for Γ-neutrality.

The enhanced portfolio is worth:

$$W(t) + n_X(t) \cdot X(t).$$

Its Γ is:

$$\Gamma_W(t) + n_X(t) \cdot \Gamma_X(t) \underset{\uparrow}{=} 0$$

for Γ-neutrality

$$\Rightarrow \quad n_X(t) = -\frac{\Gamma_W(t)}{\Gamma_X(t)}$$

The addition of X changed the Δ, however.

Q: How do we rectify this?

$\Rightarrow$ Trade in the appropriate # of shares of stock to revert to Δ-neutrality.

This does NOT change the gamma. ☺

Stock: S

$\Delta_S = \frac{\partial}{\partial S} S = 1$

$\Gamma_S = \frac{\partial}{\partial S} 1 = 0$

10. For two European call options, Call-I and Call-II, on a stock, you are given:

Greek	Call-I	Call-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746
Vega	0.0781	0.0596

Δ_I, Δ_{II}

Γ_I, Γ_{II}

Suppose you just sold 1000 units of Call-I. $\Rightarrow \boxed{n_I = -1000}$

Determine the numbers of units of Call-II and stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in Call-I.

(A) buy 95.8 units of stock and sell 872.7 units of Call-II

(B) sell 95.8 units of stock and buy 872.7 units of Call-II

(C) buy 793.1 units of stock and sell 692.2 units of Call-II

(D) sell 793.1 units of stock and buy 692.2 units of Call-II

(E) sell 11.2 units of stock and buy 763.9 units of Call-II

- n_I ... # of Call-I in the portfolio
- n_{II} ... # of Call-II —||—
- n_S ... # of shares of stock

Δ-neutral: $n_I \cdot \Delta_I + n_{II} \cdot \Delta_{II} + n_S \cdot 1 = 0$

Γ-neutral: $n_I \cdot \Gamma_I + n_{II} \cdot \Gamma_{II} = 0$

In this problem: $n_I = -1000$

$\Rightarrow n_{II} = -\frac{n_I \cdot \Gamma_I}{\Gamma_{II}} = -\frac{(-1000) \cdot 0.0651}{0.0746}$

$n_{II} > 0 \Rightarrow$ A.x , C.x

$n_{II} = 872.7 \implies$ B.

In a different problem: Also:

$n_s = -(n_I \cdot \Delta_I + n_{II} \Delta_{II}) = \ldots$

Market-Making As Insurance

- Insurance companies have two ways of dealing with unexpectedly large loss claims:

1. Hold capital reserves
2. Diversify risk by buying reinsurance

- Market-makers also have two analogous ways to deal with excessive losses:

1. Hold capital to cushion against less-diversifiable risks - When risks are not fully diversifiable, holding capital is inevitable
2. Reinsure by trading in out-of-the-money options

Continuous-time stock-price model.

Standard BM: $Z = \{Z(t), t \geq 0\}$

"infinitesimally frequent cointosses"

$\downarrow$

Arithmetic BM: $X = \{X(t), t \geq 0\}$

$$dX(t) = \mu\, dt + \sigma\, dZ(t)$$

"realized returns"

$\downarrow$

Geometric BM: $S = \{S(t), t \geq 0\}$

$$\frac{dS(t)}{S(t)} = \mu_S\, dt + \sigma_S\, dZ(t)$$

$\rightarrow$ **PRICE OPTIONS**

- focus on $\mathbb{P}^*$... risk-neutral probability measure
- build the Black-Scholes PDE which needs to be satisfied for any derivative security on S.

Continuous-time interest-rate models

$r = \{ \underline{r(t)}, t \geq 0 \}$ ... a continuous-time stochastic process denoting the short-term continuously compounded interest rate

Think about it as a non-deterministic, time-varying "force of interest".

Our models will be of the following form:

$$dr(t) = a(r(t))\,dt + \sigma(r(t))\,dZ(t)$$

DETERMINISTIC FUNCTIONS (applied to r(t))