NAME:

M339W/389W Financial Mathematics for Actuarial Applications
University of Texas at Austin
In-Term Exam II
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Notes: This is a closed book and closed notes exam. The maximum number of points on this exam is 50.
Time: 50 minutes

Problem 2.1. (5 points) Assume that the Black-Scholes setting holds. Let $\mathbf{S} = \{S(t), t \geq 0\}$ denote the price of a continuous-dividend-paying stock. The stock price today equals \$80 and its dividend yield is 0.02.

Let $r = 0.04$ be the continuously compounded risk-free interest rate.

Consider a European call option with exercise in three months and strike price $K = 80e^{0.005}$. You are given that its price today equals \$3.80.

What is the implied volatility of the stock S?

(a) 0.20
(b) 0.22
(c) 0.24
(d) 0.26
(e) None of the above.

Solution: (c)
According to the Black-Scholes formula, in our usual notation, the time-0 call price equals

$$v_C(S(0),0) = v_C(0,S(0),K,r,\delta,\sigma,T) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T}}[\ln(S(0)/K) + (r - \delta + \tfrac{1}{2}\sigma^2)T],$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Using the provided information, we obtain

$$d_1 = \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[\ln(S(0)/S(0)e^{-0.005}) + (0.04 - 0.02 + \tfrac{1}{2}\sigma^2)\frac{1}{4}\right]$$
$$= \frac{1}{\sigma\sqrt{\frac{1}{4}}}\left[-0.005 + 0.005 + \tfrac{1}{2}\sigma^2 \cdot \frac{1}{4}\right] = \frac{\sigma}{4},$$
$$d_2 = d_1 - \frac{\sigma}{2} = \frac{\sigma}{4} = -d_1.$$

Hence,

$$\begin{aligned} v_C(\ldots,\sigma) &= S(0)e^{-\delta T}N(d_1) - S(0)e^{0.005}e^{-rT}N(d_2) \\ &= 80e^{-0.005}[N(d_1) - N(-d_1)] \\ &= 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1]. \end{aligned}$$

From the problem, we know that

$$3.80 = 80e^{-0.005}[2N(\tfrac{\sigma}{4}) - 1].$$

So,

$$0.0475e^{0.005} = 2N(\tfrac{\sigma}{4}) - 1 \quad \Rightarrow \quad 2N(\tfrac{\sigma}{4}) = 1.0477381 \quad \Rightarrow \quad N(\tfrac{\sigma}{4}) = 0.523869$$

From the standard normal table, we get that

$$\tfrac{\sigma}{4} = 0.06 \Rightarrow \sigma = 0.24.$$

Problem 2.2. (5 points) The current price of a non-dividend-paying stock is \$40 per share. A market-maker writes a one-year European put option on this stock and proceeds to delta-hedge it.

The put premium is \$5.96, its delta is -0.5753, its gamma is 0.0392, and its theta is 0.01 per day.

The continuously-compounded, risk-free interest rate is 0.04.

Assuming that the stock price does not change, what is the **approximate** overnight profit for the market-maker?

(a) -0.017
(b) -0.007
(c) 0.007
(d) 0.017
(e) None of the above.

Solution: (b)
The initial cost of the total delta-hedged portfolio is

$$-5.96 + (-0.5753)(40) = -28.972.$$

The approximate put price after one day is, according to the delta-gamma-theta approximation,

$$5.96 + 0.01 = 5.97.$$

So, the overnight profit is

$$-5.97 + (-0.5753)(40) + 28.972e^{0.04/365} = -0.00682481$$

Problem 2.3. (5 points) Assume the Black-Scholes model. The current stock price is \$60 per share. The stock's dividend yield is 0.02 and its volatility is 0.3.

Consider a \$70-strike, half-year European call option on this stock. Its price is \$2.40, and its delta is 0.3.

The continuously compounded, risk-free interest rate is 0.04.

What is the volatility of the otherwise identical put option?

(a) 0.3
(b) 0.56789
(c) 1.06984
(d) 3.0563
(e) None of the above.

Solution: (c)
The volatility of the put option is

$$\sigma_P = |\Omega_P|\sigma.$$

In our usual notation, we are given that

$$\Delta_C(S(0),0) = e^{-0.02(0.5)}N(d_1(S(0),0)) = 0.3$$

On the other hand, for the put, we have

$$\begin{aligned}\Delta_P(S(0),0) &= -e^{-0.02(0.5)}N(-d_1(S(0),0))\\ &= -e^{-0.01}(1-N(d_1(S(0),0))) = -e^{-0.01} + \Delta_C(S(0),0) = -0.69005.\end{aligned}$$

By put-call parity,

$$v_P(S(0),0) = v_C(S(0),0) - S(0)e^{-\delta T} + Ke^{-rT} = 2.40 - 60e^{-0.01} + 70e^{-0.02} = 11.61.$$

So, the put's elasticity is

$$\Omega_P(S(0),0) = \frac{\Delta_P(S(0),0)S(0)}{v_P(S(0),0)} = \frac{-0.69005(60))}{11.61} = -3.56615.$$

Finally, $\sigma_P = 3.56615(0.3) = 1.06984$.

Problem 2.4. (5 points) Assume the Black-Scholes framework.

The goal is to delta-hedge a one-year, $(40,60)$-strangle on a non-dividend-paying stock whose current price is \$50. The stock's volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.10.

What is the cost of delta-hedging the strangle using shares of the underlying stock?

(a) \$16.78
(b) \$22.58
(c) \$24.33
(d) \$25.19
(e) None of the above.

Solution: (a)
The Δ of the strangle equals

$$\Delta_P(S(0), 0; K_P = 40) + \Delta_C(S(0), 0; K_C = 60),$$

i.e., it is the sum of the delta of the call with strike 60 and the delta of the put with strike 40. We have

$$d_1(S(0), 0; K_P = 40) = \frac{1}{0.2}\left[\ln\left(\frac{50}{40}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = 1.715.$$

So, the put's delta is approximately

$$-N(-d_1(S(0), 0; K_P = 40)) = -N(-1.72) = N(1.72) - 1 = -0.0427.$$

Similarly, for the call, we have

$$d_1(S(0), 0; K_C = 60) = \frac{1}{0.2}\left[\ln\left(\frac{50}{60}\right) + \left(0.10 + \frac{0.04}{2}\right)\right] = -0.31.$$

So, the call's delta is approximately

$$N(d_1(S(0), 0; K_C = 60)) = N(-0.31) = 1 - N(0.31) = 0.3783.$$

Our answer is

$$50(0.3783 - 0.0427) = 16.78.$$

Problem 2.5. A market-maker buys one option I for \$7. This option's delta is 0.5557 and its gamma is 0.01. The market maker proceeds to delta-gamma hedge this commitment by trading in the underlying and also in option II on the same stock. The latter option's price is \$3.70, its delta is -0.5794 and its gamma is 0.04.

What is the market-maker's resulting position in option II?

(a) Buy 0.25 of option II.
(b) Write 0.25 of option II.
(c) Buy 4 of option II.
(d) Write 4 of option II.
(e) None of the above.

Solution: (b)
With n_{II} denoting the position in option II, to achieve gamma-neutrality we need

$$0.01 + n_{II}(0.04) = 0 \quad \Rightarrow \quad n_{II} = -0.25.$$

Problem 2.6. Which of the following statements is always TRUE?

(a) The call rho is greater than the put rho.
(b) The put theta is always negative.
(c) The call vega is the negative of the vega of the otherwise identical put.
(d) The call psi is always positive.
(e) None of the above.

Solution: (a)

Problem 2.7. Assume the Black-Scholes model. Let the current stock price be equal to \$90 per share. Its dividend yield is 0.02 and its volatility is 0.20.

The continuously compounded, risk-free interest rate is 0.03.

Consider a one-year European call option on the above stock. The delta of this call option is 0.49. What is the strike price of the call?

(a) 84.32
(b) 87.52
(c) 90
(d) 92.74
(e) None of the above.

Solution: (d)
In our usual notation, the delta of the call is

$$\Delta_C(S(0),0) = e^{-\delta T}N(d_1(S(0),0)) = e^{-0.02}N(d_1(S(0),0)) = 0.49.$$

So,

$$N(d_1(S(0),0)) = 0.49e^{0.02} = 0.499899 \approx 0.5 \quad \Rightarrow \quad d_1(S(0),0) = 0.$$

Hence,

$$\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) = 0 \quad \Rightarrow \quad \ln\left(\frac{S(0)}{K}\right) = -\left(0.03 - 0.02 + \frac{0.04}{2}\right) = -0.03.$$

Therefore,

$$\frac{S(0)}{K} = e^{-0.03} \quad \Rightarrow \quad K = S(0)e^{0.03} = 90e^{0.03} = 92.7409.$$

Problem 2.8. Assume the Black-Scholes model. The current price of a continuous-dividend-paying stock is denoted by $S(0)$. Its dividend yield is denoted by δ and its volatility is $\sigma = 0.20$.

The continuously compounded, risk-free interest rate is equal to δ.

Consider a one-year, at-the-money European put on the above stock. What is the elasticity of this put?

(a) -0.5398
(b) -1.0473
(c) -3.5612
(d) -5.78141
(e) None of the above.

Solution: (d)
In our usual notation, since the option is at-the-money,

$$d_1(S(0),0) = \frac{1}{0.2}\left(r - \delta + \frac{(0.2)^2}{2}\right) = 0.1,$$
$$d_2(S(0),0) = d_1(S(0),0) - \sigma = 0.1 - 0.2 = -0.1.$$

So, the options delta is

$$\Delta_P(S(0),0) = -e^{-\delta}N(-d_1(S(0),0)) = -e^{-\delta}N(-0.1).$$

The put option's price is

$$\begin{aligned} v_P(S(0),0) &= S(0)e^{-r}N(-d_2(S(0),0)) - S(0)e^{-\delta}N(-d_1(S(0),0)) \\ &= S(0)e^{-r}(N(0.1) - N(-0.1)) = S(0)e^{-r}(2N(0.1) - 1). \end{aligned}$$

Hence, the elasticity is

$$\Omega_P(S(0),0) = \frac{\Delta_P(S(0),0)S(0)}{v_C(S(0),0)} = \frac{-e^{-\delta}N(-0.1)S(0)}{S(0)e^{-r}(2N(0.1)-1)} = -\frac{1-N(0.1)}{2N(0.1)-1} = -5.78141.$$

Problem 2.9. Assume the Black-Scholes model.

The continuously compounded, risk-free interest rate is 0.10.

Consider a one-year, at-the-money straddle on a non-dividend-paying stock. The stock's current price is \$50 and its volatility is 0.20.

What is the elasticity of the above straddle?

(a) 0.4514
(b) 1.1762
(c) 2.65254
(d) 6.0742
(e) None of the above.

Solution: (c)
The straddle consists of a long call and a long put, both at the money. In our usual notation, we have that

$$d_1(S(0),0) = \frac{1}{0.2}(0.10+0.02) = 0.6 \quad \text{and} \quad d_2(S(0),0) = d_1(S(0),0) - \sigma = 0.4.$$

The delta of the straddle is

$$2\Delta_C(S(0),0) - 1 = 2N(d_1(S(0),0)) - 1 = 2(0.7257) - 1 = 0.4514.$$

The Black-Scholes price of the straddle is

$$\begin{aligned} v_C(S(0),0) + v_P(S(0),0) &= S(0)N(d_1(S(0),0)) - Ke^{-r}N(d_2(S(0),0)) \\ &\quad + Ke^{-r}N(-d_2(S(0),0)) - S(0)N(-d_1(S(0),0)) \\ &= S(0)(2N(d_1(S(0),0)) - 1) - Ke^{-r}(2N(d_2(S(0),0)) - 1) \\ &= 50(0.4514 - e^{-0.1}(2(0.6554) - 1)) = 8.50883. \end{aligned}$$

Finally, the elasticity is

$$\Omega(S(0),0) = \frac{50(0.4514)}{8.50883} = 2.65254.$$

Problem 2.10. (5 points) Assume the Black-Scholes model. Let the current price of a non-dividend-paying stock be \$100. Its volatility is given to be 0.25.

The continuously compounded, risk-free interest rate is 0.02.

Under the risk-neutral probability, the probability that a one-year, European call option is in the money at expiration is 0.484.

What is the current gamma of this call option?

(a) 0.0156
(b) 0.0242
(c) 0.0288
(d) 0.0412
(e) None of the above.

Solution: (a)
We are given that, in our usual notation,

$$N(d_2(S(0),0)) = 0.484 \quad \Rightarrow \quad d_2(S(0),0) = -0.04.$$

So, $d_1(S(0),0) = -0.04 + 0.25 = 0.21$. Using the IFM formula sheet, we get

$$\Gamma_C(S(0),0) = \frac{N'(d_1(S(0),0))}{S(0)\sigma} = \frac{\frac{1}{\sqrt{2\pi}}e^{-0.21^2/2}}{100(0.25)} = 0.0156097.$$

Problem 2.11. (5 points) Assume the Black-Scholes model. Let the current stock price be \$100. Consider an option on this stock such that its current price is \$3.65, its delta is -0.4182, and its gamma is 0.016. What will the approximate price of this option be should the stock price rise to \$104 in a small time interval?

(a) 2.1052
(b) 2.0092
(c) 2.0496
(d) 1.9804
(e) None of the above.

Solution: (a)
We use the delta-gamma approximation, and get

$$\begin{aligned} v(S(dt), dt) &\approx v(S(0), 0) + \Delta(S(0), 0)(ds) + \frac{1}{2}\Gamma(S(0), 0)(ds)^2 \\ &= 3.65 + (-0.4182)(4) + \frac{1}{2}(0.016)(4)^2 = 2.1052. \end{aligned}$$

Problem 2.12. (5 points) Assume the Black-Scholes model. The price of a non-dividend-paying stock a six months ago was observed to be equal to \$40.

At that time, Bertie bought a European call option on that stock and delta-hedged his investment. At that time the call price was \$4.24 and its delta was 0.3015. Bertie never rebalanced his portfolio after the initial delta hedge. Now, Bertie wants to close his position. The current stock price is \$44 per share, the current call price is \$7.20 and the current call delta is 0.419.

The continuously compounded risk-free interest rate is 0.08.

What is Bertie's profit at the time he liquidates his portfolio?

(a) -0.10163
(b) 2.07314
(c) 1.44737
(d) 3.02185
(e) None of the above.

Solution: (b)
Since Bertie bought the call option, he needs to short shares of stock. So, the initial cost of his total delta-hedged portfolio equals

$$4.24 + (-0.3015)(40) = -7.82.$$

After half a year, Bertie's proceeds from liquidation are

$$7.20 + (-0.3015)(44) = -6.066.$$

So, his total profit is

$$-6.066 - (-7.82)e^{0.08/2} = 2.07314.$$