Multifactor Models

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Even if the market portfolio is (not) efficient, this does not mean that an efficient portfolio does not exist (think back to the tangent portfolio, e.g.).

Our purpose is that, if there is an efficient portfolio, we have for a security S:

$$\mathbb{E}[R_S] = r_f + \beta_S^{\text{eff}}(\mathbb{E}[R_{\text{eff}}] - r_f).$$

Note: Any efficient portfolio will be well-diversified.

Our "recipe" will be to construct an efficient portfolio from a collection of well-diversified portfolios.

1st Assume that we have N portfolios which can combine into an efficient portfolio; call them FACTOR PORTFOLIOS; we denote them by $F1, \dots, FN$.

2nd Q: $\mathbb{E}[R_S] = ?$ in terms of $R_{F1}, \dots, R_{FN}$ (and interactions between R_{Fi} & S).

Say that we have only two factors F1 and F2.
For the efficient portfolio built out of F1 and F2,

$$R_{\text{eff}} = w_1 \cdot R_{F1} + w_2 \cdot R_{F2}.$$

Consider the market security S.
Run the linear regression w/ two factors F1 and F2 (now it's a multiple linear regression):

$$R_S - r_f = \alpha_S + \beta_S^{F1}(R_{F1} - r_f) + \beta_S^{F2}(R_{F2} - r_f) + \varepsilon_S \sim N(0, m\varepsilon)$$ uncorrelated w/ R_{F1}, R_{F2}

Construct a portfolio P as:

- LONG security S,
- SHORT β_S^{F1} of F1,
- SHORT β_S^{F2} of F2
- <u>invest</u> the proceeds at the risk-free rate (r_f) of the short sale

$$\Rightarrow R_P = R_S - \beta_S^{F1} \cdot R_{F1} - \beta_S^{F2} \cdot R_{F2} + (\beta_S^{F1} + \beta_S^{F2}) \cdot r_f$$

$$= R_S - \beta_S^{F1}(R_{F1} - r_f) - \beta_S^{F2}(R_{F2} - r_f) \quad \text{using the linear regression}$$

$$= r_f + \alpha_S + \varepsilon_S$$

<u>Recall</u>: ε_S is uncorrelated w/ R_{F1} and R_{F2}

$\Rightarrow Cov(R_{eff}, \varepsilon_S) = 0$

$\Rightarrow \varepsilon_S$ contains just diversifiable risk

$\Rightarrow$ Risk-premium of portfolio P is zero

$\Rightarrow \boxed{\alpha_S = 0}$

<u>Attack</u> the linear regression w/ $\mathbb{E}$; get:

$$\mathbb{E}[R_S] = r_f + \beta_S^{F1}(\mathbb{E}[R_{F1}] - r_f) + \beta_S^{F2}(\mathbb{E}[R_{F2}] - r_f)$$

We can generalize it to multiple factors:

$$\mathbb{E}[R_S] = r_f + \sum_{i=1}^{N} \beta_S^{Fi}(\mathbb{E}[R_{Fi}] - r_f)$$

If all factor portfolios are self-financing:

$$\mathbb{E}[R_S] = r_f + \sum_{i}^{N} \beta_S^{Fi}\, \mathbb{E}[R_{Fi}]$$