More on Market-Making and Delta-Hedging

What do market makers do to delta-hedge?

- Recall that the delta-hedging strategy consists of **selling** one option, and **buying** a certain number Δ shares
- An example of Delta hedging for 2 days (daily rebalancing and mark-to-market):

Day 0: Share price = \$40, call price is \$2.7804, and $\Delta = 0.5824$
Sell call written on 100 shares for \$278.04, and buy 58.24 shares.
Net investment: $(58.24 \times \$40)\$278.04 = \$2051.56$
At 8%, overnight financing charge is
$\$0.45 = \$2051.56 \times (e^{-0.08/365} - 1)$

Day 1: If share price = \$40.5, call price is \$3.0621, and $\Delta = 0.6142$
Overnight profit/loss: \$29.12 \$28.17 \$0.45 = \$0.50(mark-to-market)
Buy 3.18 additional shares for \$128.79 to rebalance

Day 2: If share price = \$39.25, call price is \$2.3282

- Overnight profit/loss: \$76.78 + \$73.39 \$0.48 = \$3.87(mark-to-market)

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Self-Financing Trading: Discrete Time

- Let X_k denote the value of the hedging portfolio at time k
- Let Δ_k denote the number of shares of stock held between times k and $k+1$
- At time k, after rebalancing (i.e., moving from a position of Δ_{k-1} to a position of Δ_k), the amount we hold in the money market account is

$$X_k - S_k \Delta_k$$

- The value of the portfolio at time $k+1$ is

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k)$$

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Self-Financing Trading: Discrete Time - The Gain

- So, the gain between time k and time $k+1$ is

$$X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + r(X_k - \Delta_k S_k)$$

- This means that the gain is the sum of the capital gain from the stock holdings:

$$\Delta_k(S_{k+1} - S_k)$$

and the interest earnings from the money-market account

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Self-Financing Trading: Discrete Time - Introducing the Money-Market Account

- Define the value of a share in the money-market account at time k to be

$$M_k = (1 + r)^k$$

and let the number of shares of the money-market held at time k be denoted by Γ_k

Self-Financing Trading: Discrete Time - The new expression for the gain

- So, the gain between time k and time $k+1$ can now be written as

$$X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + \Gamma_k(M_{k+1} - M_k)$$

- Thus, the gain is the sum of the capital gain from the stock investment holdings:

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- The wealth at time $k+1$ can be expressed as

$$X_{k+1} = \Delta_k S_{k+1} + \Gamma_k M_{k+1}$$

- However, Δ_k and Γ_k cannot be chosen arbitrarily.

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Self-Financing Trading: Discrete Time - The self-financing condition

- The agent arrives at time $k+1$ with a portfolio of Δ_k shares of stock and Γ_k shares of the money market account and then rebalances, i.e., chooses Δ_{k+1} and Γ_{k+1}
- After rebalancing, the wealth of the agent is

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1}$$

- The wealth of the agent cannot be changed through rebalancing, so it must be that

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1} = \Delta_k S_{k+1} + \Gamma_k M_{k+1}$$

- The last equality yields the **discrete time self-financing condition**

$$S_{k+1}(\Delta_{k+1} - \Delta_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0$$

- The first term is the cost of rebalancing the stock and the second term is the cost of rebalancing the money-market account

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Self-Financing Trading: Segue into the continuous time

- The discrete-time self-financing condition can be rewritten as

$$S_k(\Delta_{k+1} - \Delta_k) + (S_{k+1} - S_k)(\Delta_{k+1} - \Delta_k) + M_k(\Gamma_{k+1} - \Gamma_k) + (M_{k+1} - M_k)(\Gamma_{k+1} - \Gamma_k) = 0$$

- This suggests the **continuous-time self-financing condition**

$$S_t\, d\Delta_t + dS_t\, d\Delta_t + M_t\, d\Gamma_t + dM_t\, d\Gamma_t = 0$$

- This claim can be proved using stochastic calculus ....

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Recall the meaning of Delta

- An option written on an undelying asset S is most sensitive to the changes in the value of S
- The largest part of the risk comes from the price movements of asset S - which is reflected in the delta of the option, i.e., if C is the price of a call the most pronounced effect comes from

$$\Delta_C := \frac{\partial C}{\partial S}$$

- The replicating protfolio will always contain Δ_C shares of the underlying stock
- The portfolio which contains the option, along with Δ_C shares of stock will have the value of its Delta equal to zero - we say it is **delta neutral**

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Recall the other Greeks

- If X denotes the price of a portfolio, we define

Theta:

$$\Theta := \frac{\partial X}{\partial t}$$

Gamma:

$$\Gamma := \frac{\partial^2 X}{\partial S^2}$$

Vega:

$$\mathcal{V} := \frac{\partial X}{\partial \sigma}$$

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$$\mathcal{V} := \frac{\partial X}{\partial \sigma}$$

rho:

$$\rho := \frac{\partial X}{\partial r}$$

Recall the other Greeks

- If X denotes the price of a portfolio, we define

Theta:

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The Delta-Gamma-Theta Approximation

- In this model X depends only on the values of S and t (σ and r are assumed constant)
- The Taylor expansion of X gives us

$$
\begin{aligned}
X(t+\Delta t, S+\Delta S) \\
&= X(t,s) \\
&\quad + \frac{\partial X(t,S)}{\partial S}\Delta S + \frac{\partial X(t,S)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 X(t,S)}{\partial S^2}(\Delta S)^2 \\
&\quad + \text{higher order terms}
\end{aligned}
$$

  where $\Delta S = S(t+\Delta t) - S(t)$

- Ignoring the higher order terms (as one would in establishing the Ito's Lemma), we get

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X(t+\Delta t, S+\Delta S) \\
&\approx X(t,s) + \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2}\Gamma \cdot (\Delta S)^2
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Understanding the Market-Maker's Portfolio

- Suppose that, at any time $t \leq T$, the market-maker is long $\Delta(t)$ shares of stock and short one call on that stock
- Then the cost/profit of his/her portfolio at time t equals

$$Y(t, S_t) = \Delta(t) \cdot S_t - C(t, S_t)$$

  where $C(t, S_t)$ is the price of the call at time t
- Suppose that the cost/profit above is invested in the money-market account
- Then, the change in the value of the portfolio is

$$\Delta(t) \cdot \Delta S - (C(t + \Delta t, S(t + \Delta t)) - C(t, S(t))) - r \cdot \Delta t \cdot Y(t, S_t)$$

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Understanding the Market-Maker's Portfolio: An Approximation

- The Taylor approximation similar to the one we conducted earlier yields that the change in the market-maker's portfolio is approximately

$$-\left(\frac{1}{2}(\Delta S)^2 \cdot \Gamma + \Delta t \cdot \Theta + r\Delta t \cdot (\Delta \cdot S_t - C(t, S(t)))\right)$$

  where the Greeks are calculated at time t

The Merton-Black-Scholes model

- In the Black-Scholes setting, for an option with price $C = C(t, s)$ we have that

$$(\Delta S)^2 \approx (dS)^2 = \sigma^2 S^2 \, dt$$

- So, we get

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + r(S\Delta - C) = 0$$

- Note that the value of Θ is fixed once we fix the values of Δ and Γ
- Note that Γ is the measure of risk the hedger faces as a result of not rebalancing frequently enough
- so, in the absence of **transaction costs**, the agent should rebalance often as this reduces the variance of his/her portfolio

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- The recipe for Delta-hedging is the same as for the European option, but restricted to the **continuation region**, i.e., the region prior to early exercise (if it is to happen)
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Delta-Hedging in Practice

- As we have seen thus far, the market-maker may adopt a **Delta-neutral** position to try to make her portfolio less sensitive to uncertainty, i.e., the changes in S
- Alternatively, one may adopt a **Gamma-neutral** position by using options to hedge - it is necessary to use other types of options for this strategy
- Augment the portfolio by by buying deep-out-of-the-money options as insurance - this is probably not a viable strategy
- Use **static option replication** according to put-call parity to form a both Delta- and Gamma-neutral hedge
- A relatively novel approach involves trading the hedging error as another **financial product**

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Gamma-Neutrality

- Denote by Γ the gamma of a certain portfolio X and by Γ_C the gamma of a certain contingent claim C.
- In addition to having X, we want to buy/sell n contracts of C in order to make the entire portfolio gamma-neutral, i.e., we want to have

$$\Gamma + n\Gamma_C = 0$$

- So, the correct number of contingent claims C to buy/sell is

$$n = -\frac{\Gamma}{\Gamma_C}$$

- However, this addition to our position changes the delta of the entire portfolio
- To rectify this, we trade a certain (appropriate) number of the shares of the underlying asset to make the entire portfolio delta-neutral, as well
- Note that the last step does not alter the gamma of the entire portfolio, as the gamma of the underlying stock is always equal to zero

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Gamma-Neutrality: An Example

- Consider a Delta-neutral portfolio X that has $\Gamma = -5,000$
- Assume that the traded option has $\Delta_C = 0.4$ and $\Gamma_C = 2$
- We offset the negative gamma of the portfolio X by purchasing $n = 5,000/2 = 2,500$ option contracts
- The resulting portfolio is gamma-neutral, but has the delta equal to

$$\Delta_{new} = 2,500\Delta_C = 2,500 \cdot 0.4 = 1,000$$

- To rectify this, we should sell 1,000 shares of the underlying asset
- Similarly, one should be able to make one's portfolio neutral with respect to the other Greeks if there is nontrivial dependence of the portfolio on the relevant parameter (r or σ, e.g.)

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Market-Making As Insurance

- Insurance companies have two ways of dealing with unexpectedly large loss claims:

1. Hold capital reserves
2. Diversify risk by buying reinsurance

- Market-makers also have two analogous ways to deal with excessive losses:

1. Hold capital to cushion against less-diversifiable risks - When risks are not fully diversifiable, holding capital is inevitable
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