Log-Normal Stock Prices

M339W: February 10th, 2020.

$S(T)$ ... time-T stock price

$R(0,T)$ ... realized return, i.e.,

$$R(0,T) = \ln\left(\frac{S(T)}{S(0)}\right) \iff S(T) = S(0)e^{R(0,T)}$$

We model $R(0,T)$ as normally dist'd, i.e.,

$$R(0,T) \sim Normal(mean = m, var = v^2 = \sigma^2 \cdot T)$$

$\Rightarrow$ $S(T)$ is lognormally dist'd

$$\Rightarrow \mathbb{E}[S(T)] = S(0)e^{m + \frac{1}{2}\sigma^2 T} \qquad (LD)$$

Market Model

- RISKLESS asset w/ the continuously compounded, risk-free i.r. r
- RISKY asset: for now a continuous-dividend paying stock; we have
  - δ ... dividend yield
  - σ ... volatility
  - α ... (mean) rate of return, i.e.,

a constant such that the expected wealth of an investor in one share at time-0 equals

$$S(0)e^{\alpha \cdot T} \quad \text{at time-}T$$

0 —————————— T

buy <u>one</u> share

own $\underline{e^{\delta \cdot T}}$ shares
w/ each worth $S(T)$
$\Rightarrow$ the expected wealth is

$$\mathbb{E}\left[e^{\delta \cdot T} \cdot S(T)\right] = e^{\delta \cdot T}\,\mathbb{E}[S(T)]$$

We get that, in our model, α must satisfy

$$S(0)\, e^{\alpha \cdot T} = e^{\delta \cdot T} \cdot \mathbb{E}[S(T)]$$

(mean) rate of appreciation

$$\Rightarrow \quad \mathbb{E}[S(T)] = S(0)\, e^{(\alpha - \delta) \cdot T} \qquad (M)$$

We equate (LD) & (M), and we get

$$\mathbb{E}[S(T)] = S(0)\, e^{m + \frac{\sigma^2 \cdot T}{2}} = S(0)\, e^{(\alpha - \delta) \cdot T}$$

$$\Rightarrow \quad m = ?$$

$$m + \frac{\sigma^2 \cdot T}{2} = (\alpha - \delta) \cdot T$$

$$\Rightarrow \quad m = \underbrace{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)}_{=:\, \mu} \cdot T$$

↑ shorthand

2.

$\Rightarrow R(0,T) \sim N\left(\text{mean} = \left(\alpha - \delta - \frac{\sigma^2}{2}\right) \cdot T,\ \text{var} = \sigma^2 \cdot T\right)$

<u>Then</u>, $R(0,T)$ can be expressed as a linear transform of a standard normal $Z \sim N(0,1)$, i.e.,

$$R(0,T) = \left(\alpha - \delta - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z$$

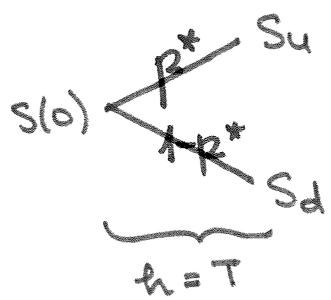
$$\Rightarrow \quad S(T) = S(0)\, e^{\left(\alpha - \delta - \frac{\sigma^2}{2}\right) \cdot T + \sigma\sqrt{T} \cdot Z}$$

On Pricing.

When our focus is the quality of investment, say, in terms of the expected payoff (of the stock, or an option on the stock), we look @ the PHYSICAL/SUBJECTIVE PROBABILITY, i.e., the one using the parameter α.

Q: What do we change in our model to be able to price?

Recall: In the one-period binomial tree:



q: What is, under the risk-neutral probability, your expected wealth @ time T?

$$\mathbb{E}^*\left[e^{\delta \cdot T} \cdot S(T)\right] =$$

$$= e^{\delta \cdot T} \cdot \mathbb{E}^*[S(T)]$$

$$= e^{\delta \cdot T}\left(p^* \cdot S_u + (1-p^*) \cdot S_d\right)$$

$$= e^{\delta \cdot T}\left(\frac{e^{(r-\delta)h} - d}{u-d} \cdot u \cdot S(0) + \frac{u - e^{(r-\delta)h}}{u-d} \cdot d \cdot S(0)\right)$$

$$= e^{\delta \cdot T} \cdot S(0) \cdot \frac{ue^{(r-\delta)h} - \cancel{ud} + \cancel{ud} - de^{(r-\delta)h}}{u-d}$$

$$= e^{\delta \cdot T} \cdot S(0) \cdot \frac{e^{(r-\delta)h}\cancel{(u-d)}}{\cancel{u-d}}$$

$$\underset{h=T}{=} S(0)\, e^{r \cdot T}$$

Note:

$$\mathbb{E}^*[S(T)] = S(0)\, e^{(r-\delta)\cdot T} = F_{0,T}(S)$$

UNDER THE RISK-NEUTRAL MEASURE, r PLAYS THE ROLE of the mean RATE OF RETURN.

In the continuous model, we have that under the risk-neutral probability measure $\mathbb{P}^*$ the realized returns can be modeled as:

$$R(0,T) \sim N\left(\text{mean} = \left(r - \delta - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)$$

We use this law to price options!

$\Rightarrow$ Under $\mathbb{P}^*$, w/ $Z \sim N(0,1)$

$$S(T) = S(0)e^{\left(r - \delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

Q: What is the expected time-T stock price under $\mathbb{P}^*$?

$\rightarrow$: $$\mathbb{E}^*[S(T)] = S(0)e^{(r-\delta)\cdot T} = \underbrace{F_{0,T}(S)}_{\text{the forward price}}$$

:)