M339W: February 12th, 2020

Log-Normal Stock Prices

aka (in the SoA problems)
the Black-Scholes model;
the geometric Brownian motion

Review:

$$R(0,T) \sim N\left(\text{mean} = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T,\ \text{var} = \sigma^2 \cdot T\right)$$

$\Rightarrow$ w/ $Z \sim N(0,1)$

we can express the stock prices as

$$S(T) = S(0)\, e^{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

$$\Rightarrow \mathbb{E}[S(T)] = S(0)\, e^{(\alpha-\delta)\cdot T}$$

Q: What is the median time-T stock price?

$\rightarrow$: $S(0)\, e^{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T}$

Q: Given the mean & the median of the time-t stock price, what is the volatility?

$$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}\, e^{\cancel{(\alpha-\delta)\cdot T}}}{\cancel{S(0)}\, e^{(\cancel{\alpha-\delta}-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2 T}{2}}$$

$$\Rightarrow \ln\left(\frac{\text{mean}}{\text{median}}\right) = \frac{\sigma^2 T}{2}$$

$$\Rightarrow \sigma = \sqrt{\frac{2}{T}\ln\left(\frac{\text{mean}}{\text{median}}\right)}$$

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Problem Set # 3

Mean and median of the log-normal stock prices.

Problem 3.1. The current price of a continuous-dividend-paying stock is \$80 per share. Its rate of appreciation is 12% and its volatility is 30%.

Let $R(0,t)$ denote the realized return of this stock over the time period $[0,t]$ for any $t > 0$. Calculate $\mathbb{E}[R(0,2)]$.

volatility

$\rightarrow$: $R(0,T) \sim N(\text{mean} = (\alpha - \delta - \frac{\sigma^2}{2})\cdot T, \ \text{var} = \sigma^2 \cdot T)$

rate of appreciation

$$\mathbb{E}[R(0,2)] = \left(0.12 - \frac{(0.3)^2}{2}\right)\cdot 2 = (0.12 - 0.045)\cdot 2 = 0.15$$

Problem 3.2. (5 points) S(0) $\alpha - \delta$

A stock is valued at \$75.00. The annual expected rate of appreciation is 10.0% and the standard deviation of annualized returns is 25.0%. If the stock is lognormally distributed, what is the expected stock price after 2 years? σ

(a) About \$71.61
(b) About \$81.63
(c) About \$91.61
(d) About \$108.83
(e) None of the above.

$$\mathbb{E}[S(2)] = S(0)e^{(\alpha-\delta)\cdot 2}$$
$$= 75 \cdot e^{0.1\cdot(2)} = 75e^{0.2}$$
$$\approx 91.605$$

Problem 3.3. (5 pts) A non-dividend-paying stock is valued at \$55.00 per share. The annual expected (rate of) return is 12.0% and the standard deviation of annualized returns is given to be 22.0%. If the stock price is modeled using the lognormal distribution (as discussed in class), what is the median of the stock price in 3 years?

$S(0)$

(a) \$57.67
(b) \$67.67
(c) \$73.31
(d) \$87.31
(e) None of the above.

$\alpha = 0.12, \sigma = 0.22$

$$S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot T} = 55e^{\left(0.12 - 0 - \frac{(0.22)^2}{2}\right)\cdot 3}$$
$$\approx 73.31 \Rightarrow (c)$$

2.

Problem 3.4. Assume that the stock price is modeled using the lognormal distribution. The annual mean rate of appreciation on the stock is given to be 12%. The median time$-t$ stock price is evaluated to be $S(0)e^{0.1t}$. What is the volatility parameter of this stock price?

$\alpha - \delta = 0.12$; median time$\cdot t$ stock price

$$= S(0)e^{0.1t} = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot t}$$

$$\Rightarrow \quad 0.12 - \frac{\sigma^2}{2} = 0.1 \Rightarrow \frac{\sigma^2}{2} = 0.02 \Rightarrow \sigma^2 = 0.04$$

$$\Rightarrow \boxed{\sigma = 0.2}$$

Problem 3.5. The current stock price is \$100 per share. The stock price at any time $t > 0$ is modeled using the lognormal distribution. Assume that the continuously compounded mean rate of return for the stock equals 12%. Let the stock's dividend yield be 4% and let its volatility equal 20%.

Find the value t^* at which the median stock price equals \$120.

$S(0) = 100$; $\alpha = 0.12$, $\delta = 0.04$, $\sigma = 0.20$

Condition : $S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot t^*} = 120 \quad / : S(0)$

$$e^{(0.12-0.04-\frac{0.04}{2})\cdot t^*} = 1.2$$

$$\Rightarrow \quad 0.06 \cdot t^* = \ln(1.2)$$

$$\Rightarrow \quad t^* = 3.0387$$

Problem 3.6. The volatility of the price of a continuous-dividend-paying stock is 0.2. The stock price is modeled using a log-normal distribution. The expected time-2 stock price is \$120.

Then, the median of the time-2 stock price falls within this interval:

(a) $[0, 86)$
(b) $[86, 106)$
(c) $[106, 112)$
(d) $[112, 124)$
(e) None of the above.

$\sigma = 0.2$

$\mathbb{E}[S(2)] = 120$

$$\text{median} = S(0)e^{(\alpha-\delta-\frac{\sigma^2}{2})\cdot 2}$$

$$= S(0)e^{(\alpha-\delta)\cdot 2} \cdot e^{-\frac{\sigma^2}{\cancel{2}}\cdot\cancel{2}}$$

$$= 120 \cdot e^{-0.04} = 115.295$$

3.

Risk Measures.

The Variance.

For any random variable X, we have the expected value of X as

$$\mu_X := \mathbb{E}[X] \quad \text{(if it exists)}$$

We define the variance of X as

$$\mathrm{Var}[X] := \mathbb{E}\left[(X-\mu_X)^2\right] = \mathbb{E}[X^2] - \mu_X^2$$

(if it exists)

Usage: $X \leftrightarrow R$ ... return on an investment

The Semi-Variance.

Define the semi-variance of X as:

$$\sigma_{SV}^2 := \mathbb{E}\left[\left(\min(0, X-\mu_X)\right)^2\right]$$

Value @ Risk.

α ... probability of an adverse event you're still willing to live with.

R ... return random variable, i.e., we benefit if R is high and we have an adverse effect if R is low

Define. $VaR_\alpha(R)$ as the value π_α such that:

$$\mathbb{P}[R \le \pi_\alpha] = \alpha$$

Temporarily assume that R is a continuous random variable; the graph of its density is