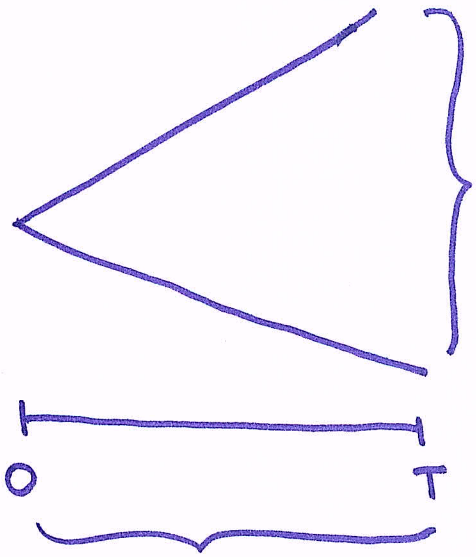
02/16/2018.

Happy New Year!

Binomial tree for the stock price:



Hoping to have a "large" n !

Environment:

- r ... cont. comp., RISK-FREE interest rate
- $S(0)$ ... initial stock price
- δ ... dividend yield
- α .... mean rate of return on the stock (on the annual scale) (continuously compounded)
- σ ... volatility of the stock
- 

TIME HORIZON (say, the expiration date of an option)

$\Rightarrow$ For n periods: the length of the period

$$h_n = \frac{T}{n}$$

For the <u>forward tree</u>:

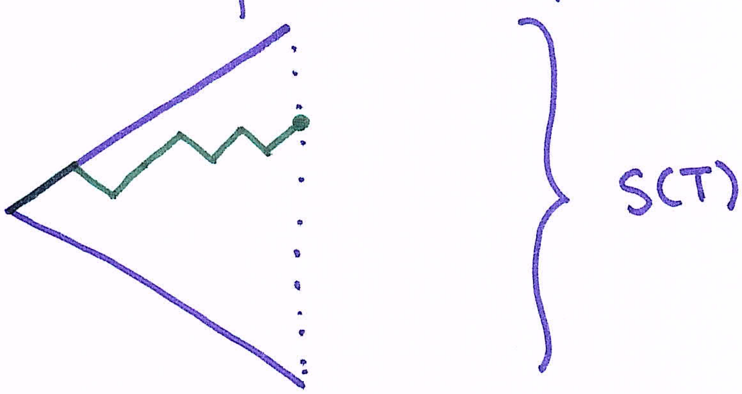
$$u_n = e^{(r-\delta)h_n + \sigma\sqrt{h_n}}$$

$$d_n = e^{(r-\delta)h_n - \sigma\sqrt{h_n}}$$

On the other hand: our "subjective" probab. associated w/ α

$$p_n = \frac{e^{(\alpha-\delta)\cdot h_n} - d_n}{u_n - d_n}$$

Q: With the above u_n and d_n, what is the "expression" for $S(T)$ for a particular n?



$$S(T) = S(0)\cdot u_n^{\#\text{ of steps up}} \cdot d_n^{\#\text{ of steps down}}$$

$$\underbrace{\#\text{ of steps up}}_{X_n} + \#\text{ of steps down} = n$$

$$X_n \sim \text{Binomial}(n, p_n)$$

$$\Rightarrow S(T) = S(0)\cdot u_n^{X_n} \cdot d_n^{n-X_n}$$

$$S(T) = S(0)\left(\frac{u_n}{d_n}\right)^{X_n} \cdot d_n^n$$

$$= S(0)\left(e^{2\sigma\sqrt{h_n}}\right)^{X_n} \cdot e^{((r-\delta)\cdot h_n + \sigma\cdot\sqrt{h_n})\cdot n}$$

$$= S(0)\, e^{2\sigma \frac{\sqrt{T}}{\sqrt{n}}\cdot X_n} \cdot e^{(r-\delta)\cdot T - \sigma\cdot\frac{\sqrt{T}}{\cancel{\sqrt{n}}}\cdot\frac{\sqrt{n}}{\cancel{n}}}$$

<u>Recall:</u>

<u>SLLL</u> (The Strong Law of Large Numbers)

A sequence $\{Y_n, n \geq 1\}$ of independent, identically distributed rnd variables such that

$$\mu_Y := \mathbb{E}[Y_1] < \infty$$

Then,
$$\frac{Y_1 + \dots + Y_n}{n} \xrightarrow[n\to\infty]{} \mu_Y \quad .$$

<u>Also:</u> Let g be a function such that $g(Y_1)$ is well defined and $\mathbb{E}[g(Y_1)] < \infty$

Then:
$$\frac{g(Y_1) + \dots + g(Y_n)}{n} \xrightarrow[n\to\infty]{} \mathbb{E}[g(Y_1)]$$

CLT (The Central Limit theorem aka the normal approximation)

A sequence $\{Y_n, n \geq 1\}$ of i.i.d. rnd vars such $\mu_Y := \mathbb{E}[Y_1] < \infty$ and $\sigma_Y^2 := Var[Y_1] < \infty$.

Then,
- Set $\bar{Y}_n = \frac{1}{n}(Y_1 + \cdots + Y_n)$
- Note: $\mathbb{E}[\bar{Y}_n] = \mu_Y$;

$$Var[\bar{Y}_n] = \frac{\sigma_Y^2}{n} \Rightarrow SD[\bar{Y}_n] = \frac{\sigma_Y}{\sqrt{n}}$$

we have:

$$\frac{\bar{Y}_n - \mu_Y}{\frac{\sigma_Y}{\sqrt{n}}} \underset{n\to\infty}{\Longrightarrow} N(0,1)$$

Random Walks.: Good model for paths that your stock price takes through the tree.

Consider a sequence of i.i.d. rand vars:

$$\xi_k \sim \begin{cases} +1 & \text{w/ probab. } p \quad \ldots \text{ step } \underline{\text{up}} \text{ in the tree} \\ -1 & \text{w/ probab. } 1-p \quad \ldots \text{ step } \underline{\text{down}} \text{ in the tree} \end{cases}$$

for $k = 1, 2, \ldots\ldots$

We define: $X(0) = 0$;

$$X(n) = \sum_{k=1}^{n} \xi_k = X(n-1) + \xi_n; \quad n = 1, 2, \ldots$$

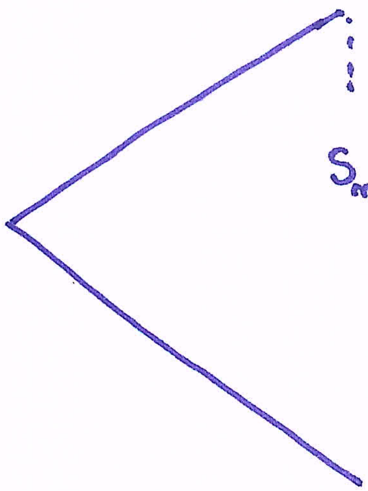
This process is called a SIMPLE RANDOM WALK.

If $p = \frac{1}{2}$, the it's a SYMMETRIC R.W.

$X(n)$ ... position / height in the stock-price tree

$$X(n) = \sum_{k=1}^{n} \xi_k = \underbrace{1 \cdot \sum_{\xi_k = +1} \xi_k - 1 \cdot \sum_{\xi_k = -1} |\xi_k|}$$

$= \#$ of steps up $- \#$ of steps down



$S_{m,k}$

k ... # of steps up − # of steps down

$$S(T) = S(0) \left(\frac{u_n}{d_n}\right)^{\#\text{ of upsteps}} \cdot d_n^n$$

$$+ \begin{cases} \#\text{ of upsteps} - \#\text{ of downsteps} = \underset{\text{rnd walk position}}{\underbrace{X(n)}} \\ \#\text{ of upsteps} + \#\text{ of downsteps} = n \end{cases}$$

$$\#\text{ of upsteps} = \frac{1}{2}\left(\underline{\underline{X(n)}} + n\right)$$