M339W: February 19th, 2020.

Value@Risk (VaR) [review].

Start w/ a (small) probability α.

- If your random variable R corresponds to a return or a profit, the the adverse event contains small values of the random variable R.

  The $VaR_\alpha(R)$ is a number such that

  $$\mathbb{P}[R \leq VaR_\alpha(R)] = \alpha.$$

- If X corresponds to a loss, we set $VaR_\alpha(X)$ to be a constant such that

  $$\mathbb{P}[X > VaR_\alpha(X)] = \alpha$$

$\delta = 0$

35) You own a share of a nondividend-paying stock and will hold it for a period of time. You want to set aside an amount of capital as a percentage of the initial stock price to reduce the risk of loss at the end of the holding period.

You are given: $\hookrightarrow \varphi \cdot S(0) = C$... the amt of capital set aside

i) The stock price follows a lognormal distribution.
ii) The annualized expected rate of return on the stock is 15%. $\alpha = 0.15$
iii) The annualized stock volatility is 40%. $\sigma = 0.40$
iv) The investment period is 4 years. $T = 4$
v) The Value-at Risk (VaR) at the 3rd percentile for the capital plus the ending stock value equals the initial stock price.

Calculate the capital amount as a percentage of initial stock price.

(A) 57%

(B) 63%

(C) 71%

(D) 82%

(E) 91%

(v): The requirement is

$$VaR_{0.03}(S(T) + C) = S(0)$$

By the def'n of VaR:

$$\mathbb{P}[S(T) + \underset{\varphi \cdot S(0)}{C} \leq S(0)] = 0.03$$

$$S(T) = S(0) e^{(\alpha - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

$$\mathbb{P}[\cancel{S(0)} e^{(0.15 - \frac{(0.4)^2}{2}) \cdot 4 + 0.4\sqrt{4} \cdot Z} \leq \cancel{S(0)}(1-\varphi)] = $$

$$= 0.03$$

Find $z^*_{0.03} = N^{-1}(0.03) = -N^{-1}(0.97) = -1.88$

$$\Rightarrow 1 - \varphi = e^{(0.15 - \frac{0.16}{2}) \cdot 4 + 0.4(2)(-1.88)}$$

$$\Rightarrow \varphi = 1 - e^{-1.224} = 0.7059 \Rightarrow (C)$$

■ 2.

Tail Value @ Risk (TVaR).

Start w/ a (small) probability α.

- Let R correspond to a return or profit or a stock price, then the TVaR is defined as:

$$TVaR_{\alpha}(R) := \mathbb{E}\left[R \mid R \leq VaR_{\alpha}(R)\right]$$

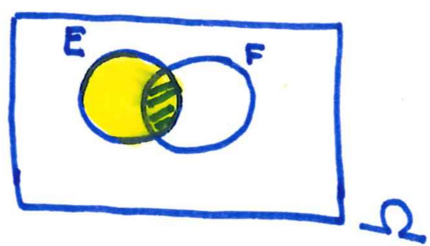
Recall: $\mathbb{P}[R \leq \underbrace{VaR_{\alpha}(R)}_{\text{a number}}] = \alpha$ (assume R continuous)

For the $TVaR_{\alpha}(R)$, we have:

$$\mathbb{E}\left[R \mid R \leq VaR_{\alpha}(R)\right] = ?$$

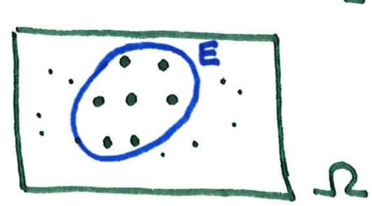
For conditional probability:
Consider two events E & F such that $\mathbb{P}[E] > 0$, we define

$$\mathbb{P}[F \mid E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]}$$



For conditional expectation:
Consider a random variable X & an event E such that $\mathbb{P}[E] > 0$,

$$\mathbb{E}[X \mid E] = ?$$



$$\mathbb{E}[X \mid E] = \frac{\mathbb{E}[X \cdot \mathbb{I}_E]}{\mathbb{P}[E]}$$

Within the context of TVaR, we take

$$E = \{ R \leq VaR_\alpha(R) \}.$$

We get:

$$TVaR_\alpha(R) = \frac{\mathbb{E}[R \cdot \mathbb{I}_{[R \leq VaR_\alpha(R)]}]}{\underbrace{\mathbb{P}[R \leq VaR_\alpha(R)]}_{\alpha}} =$$

$$= \frac{1}{\alpha} \mathbb{E}[R \cdot \mathbb{I}_{[R \leq VaR_\alpha(R)]}]$$

If R is continuous, w/ density f_R, <u>then</u>

$$\boxed{TVaR_\alpha(R) = \frac{1}{\alpha} \int_{-\infty}^{VaR_\alpha(R)} x \cdot f_R(x)\, dx}$$

- Let X correspond to a loss, say a severity random variable. <u>Then</u>,

$$TVaR_\alpha(X) = \mathbb{E}[X \mid X > VaR_\alpha(X)]$$

$= \ldots$ perform the same steps as above.

Coherent risk measures.

Usually denoted by ρ (but, the SoA uses g).
Let X represent a loss.

(I) Translational Invariance

for a positive constant c, we require

$$\rho(X+c) = \rho(X) + c$$

(II) Positive Homogeneity

for a positive constant c, we require

$$\rho(c \cdot X) = c \cdot \rho(X)$$

(III) Subadditivity

for X and Y loss random variables, we require

$$\rho(X+Y) \leq \rho(X) + \rho(Y)$$

(IV) Monotonicity

for X and Y representing losses, we require:

IF $X \leq Y$ [i.e., $X \leq Y$ almost surely, i.e., $\mathbb{P}[X \leq Y] = 1$]

THEN $\rho(X) \leq \rho(Y)$