Partial and conditional expectations.

- Motivation I : $TVaR_{\alpha}(S(T))$
- Motivation II : $\boxed{\text{PRICING}}$

Goal: Prices of European options on a stock modeled in the Black-Scholes framework.

Idea: Use risk-neutral pricing:

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\substack{\text{the payoff}\\ \text{of our European option}}}]}$$

Temporarily: Focus on a time-T, strike-K call option on our log-normal stock:

The payoff @ time-T :

$$V_c(T) = (S(T)-K)_+$$

$$\Rightarrow \mathbb{E}[V_c(T)] = \mathbb{E}[(S(T)-K)_+]$$

$$= \mathbb{E}[(S(T)-K)\cdot \mathbb{I}_{[S(T)\geq K]}]$$

(the linearity of $\mathbb{E}$) $$= \mathbb{E}[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}] - \mathbb{E}[K\cdot \mathbb{I}_{[S(T)\geq K]}]$$

1.

$$= \mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}\right] - K\,\underbrace{\mathbb{E}\left[\mathbb{I}_{[S(T)\geq K]}\right]}$$

partial expectation
???

$$\mathbb{I}_{[S(T)\geq K]} = \begin{cases} 1 & \text{if } \{S(T)\geq K\} \\ 0 & \text{if not} \end{cases}$$

$$\Rightarrow \mathbb{E}\left[\mathbb{I}_{[S(T)\geq K]}\right] =$$

$$= 1\cdot \mathbb{P}[S(T)\geq K] + 0\cdot \mathbb{P}[S(T)<K]$$

$$= \mathbb{P}[S(T)\geq K] = N(\hat{d}_2)$$

$$\text{w/ } \hat{d}_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha-\delta-\frac{\sigma^2}{2}\right)\cdot T\right]$$

Recall: $S(T) = S(0)\,e^{\overbrace{(\alpha-\delta-\frac{\sigma^2}{2})}^{=:\mu}\cdot T + \sigma\sqrt{T}\cdot Z}$, $Z\sim N(0,1)$

Focus on $\mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}\right]$.

Idea: Use the formula for the expectation of a function of a random var. Z.

Note: $\{S(T)\geq K\} \Leftrightarrow$

$$\Leftrightarrow \left\{S(0)\,e^{\mu\cdot T + \sigma\sqrt{T}\cdot Z} \geq K\right\}$$

$$\Leftrightarrow \left\{Z > -\hat{d}_2\right\}$$

z ... independent argument which serves as the placeholder for Z

Set: $g(z) = S(0)\, e^{\mu \cdot T + \sigma\sqrt{T} \cdot z}$

(so that $g(Z) = S(T)$).

$\Rightarrow$ Our partial expectation is

$$\mathbb{E}\left[g(Z) \cdot \mathbb{I}_{[Z > -\hat{d}_2]}\right] =$$

$$= \int_{-\hat{d}_2}^{+\infty} g(z)\, \varphi(z)\, dz$$ ... fill in the algebra ...

$$= \overbrace{S(0)\, e^{(\alpha - \delta)\cdot T}}^{=: \mathbb{E}[S(T)]} \cdot N(\hat{d}_1)$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$

Consequently, the expected payoff of our call option is:

$$\mathbb{E}[S(T)] \cdot N(\hat{d}_1) - K \cdot N(\hat{d}_2)$$

For the put option, the following expression is relevant:

$$\mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)<K]}\right] =$$

$$= \mathbb{E}[S(T)] - \mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)\geq K]}\right]$$

$$= S(0)e^{(\alpha-\delta)\cdot T} - S(0)e^{(\alpha-\delta)\cdot T}\cdot N(\hat{d}_1)$$

$$= S(0)e^{(\alpha-\delta)\cdot T}\left(1-N(\hat{d}_1)\right)$$

$$= S(0)e^{(\alpha-\delta)\cdot T}\cdot N(-\hat{d}_1)$$

<u>Conditional Expectations</u>:

- $$\mathbb{E}[S(T) \mid S(T)>K] = \frac{\mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)>K]}\right]}{\mathbb{P}[S(T)>K]}$$

$$= \frac{S(0)e^{(\alpha-\delta)\cdot T}\cdot N(\hat{d}_1)}{N(\hat{d}_2)}$$

- $$\mathbb{E}[S(T) \mid S(T)<K] = \frac{\mathbb{E}\left[S(T)\cdot \mathbb{I}_{[S(T)<K]}\right]}{\mathbb{P}[S(T)<K]}$$

$$= \frac{S(0)e^{(\alpha-\delta)\cdot T}\cdot N(-\hat{d}_1)}{N(-\hat{d}_2)}$$

Note. For a probability level $p \in (0,1)$, we can set $K = VaR_p(S(T))$

$$TVaR_p(S(T)) \underset{\substack{\uparrow \\ \text{by def'n}}}{=} \mathbb{E}\left[S(T) \mid S(T) < VaR_p(S(T))\right]$$

$$= \frac{\mathbb{E}\left[S(T) \cdot \mathbb{I}_{[S(T) < VaR_p(S(T))]}\right]}{p} \dots$$

Black-Scholes Pricing:

In general, under any measure $\mathbb{P}$,

$$\mathbb{E}[V_c(T)] = S(0)e^{(\alpha-\delta)\cdot T} N(\hat{d}_1) - K \cdot N(\hat{d}_2)$$

The risk-neutral pricing principle:

$$V(0) = e^{-rT}\,\mathbb{E}^*[V(T)]$$

In the Black-Scholes model, under the risk-neutral probab. measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r-\delta-\frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

$\Rightarrow$ We get, for the call option:

$$V_c(0) = e^{-rT}\,\mathbb{E}^*[V_c(T)]$$

$$= e^{-rT}\Big(S(0)e^{(r-\delta)\cdot T} \cdot N(d_1) - K \cdot N(d_2)\Big)$$

$T = 1/4 \quad K = 25$

* **6.** You are considering the purchase of 100 units of a 3-month 25-strike European call option on a stock.

You are given:

(i) The Black-Scholes framework holds.
(ii) The stock is currently selling for 20. $S(0) = 20$
(iii) The stock's volatility is 24%. $\sigma = 0.24$
(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\delta = 0.03$
(v) The continuously compounded risk-free interest rate is 5%. $r = 0.05$

Calculate the price of the block of 100 options.

Usual steps:

(A) 0.04
(B) 1.93 1^{st} d_1 & d_2
(C) 3.63 2^{nd} $N(d_1)$ & $N(d_2)$
(D) 4.22
(E) 5.09 3^{rd} $V_c(0) = B\cdot S$ Price

7. Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.

(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.

(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.

(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.

(vii) 1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

6.

$\Rightarrow$

$$\boxed{\begin{gathered} V_c(0) = S(0)e^{-\delta\cdot T} N(d_1) - Ke^{-r\cdot T} N(d_2) \\ \text{w/ } d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right] \\ \text{and} \\ d_2 = d_1 - \sigma\sqrt{T} \end{gathered}}$$

THE BLACK·SCHOLES CALL PRICE.

Note: · $S(0)e^{-\delta\cdot T} = F^P_{0,T}(S)$

· $Ke^{-rT} = PV_{0,T}(K)$

Next: What about the put price?

By put·call parity:

$$V_c(0) - V_p(0) = F^P_{0,T}(S) - PV_{0,T}(K)$$

$$\Rightarrow V_p(0) = V_c(0) - F^P_{0,T}(S) + PV_{0,T}(K)$$

$$= S(0)e^{-\delta\cdot T}\cdot N(d_1) - S(0)e^{-\delta T} - Ke^{-rT}N(d_2) + Ke^{-rT}$$

$$= Ke^{-rT}\big(\underbrace{1 - N(d_2)}_{N(-d_2)}\big) - S(0)e^{-\delta T}\big(\underbrace{1 - N(d_1)}_{N(-d_1)}\big)$$

$$\boxed{\Rightarrow V_p(0) = Ke^{-rT}\cdot N(-d_2) - S(0)e^{-\delta\cdot T}N(-d_1)}$$

The Black·Scholes Put Price

$1^{\underline{st}}$ $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$

$$d_1 = \frac{1}{0.24\sqrt{1/4}}\left[\ln\left(\frac{20}{25}\right) + \left(0.05 - 0.03 + \frac{(0.24)^2}{2}\right)\cdot\frac{1}{4}\right]$$

$\Rightarrow d_1 = -1.76$

$\Rightarrow d_2 = d_1 - \sigma\sqrt{T} = -1.76 - 0.24\sqrt{1/4} = -1.88$

$2^{\underline{nd}}$ $N(d_1) = N(-1.76) = 1 - N(1.76)$

$= 1 - 0.9608 = \underline{0.0392}$;

$N(d_2) = N(-1.88) = 1 - N(1.88)$

$= 1 - 0.9699 = \underline{0.0301}.$

$3^{\underline{rd}}$

$$V_c(0) = 20e^{-0.03\left(\frac{1}{4}\right)}\cdot 0.0392 - 25e^{-0.05\left(\frac{1}{4}\right)}\cdot 0.0301$$

$\Rightarrow V_c(0) = 0.03499$

$\Rightarrow$ answer : $100\cdot V_c(0) = 3.499$

Discrepancy due to std normal tables.

:(

<u>Final answer:</u>

3.63.

8.