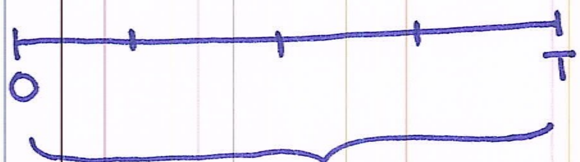
Realized Returns

Recall:



$n \ldots$ # of periods

$\Rightarrow$ length of each period : $h_n = \frac{T}{n}$

$\Rightarrow$ u_n, d_n factors

and $p_n (= p_n^*)$ dependent on n

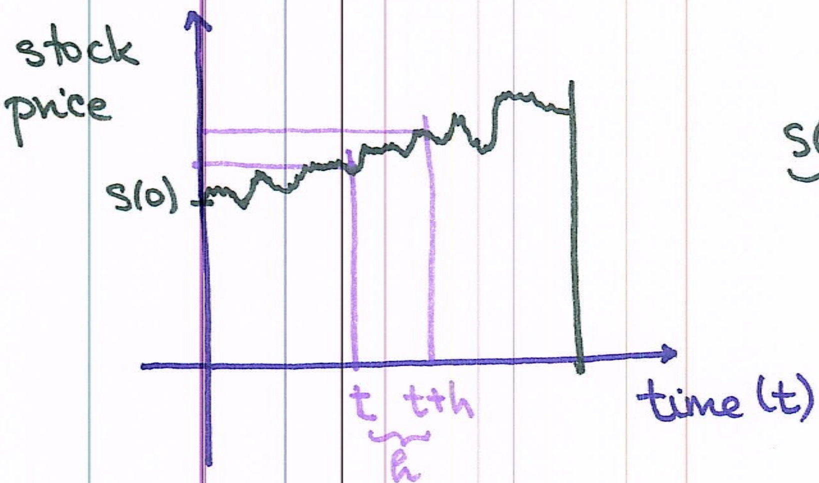
Properties of our model:

- realized returns are independent between periods;
- realized returns are identically dist'd over different periods of the same length

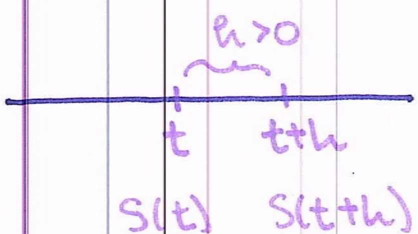
$\rightarrow$ We want to carry over these properties into the continuous model.

$\boxed{\text{M339W: February } 5^{th}, \text{ 2020.}}$

Stock prices in continuous time: stochastic processes



$\underbrace{S(t)}$, $t \geq 0$ ← time t stock price

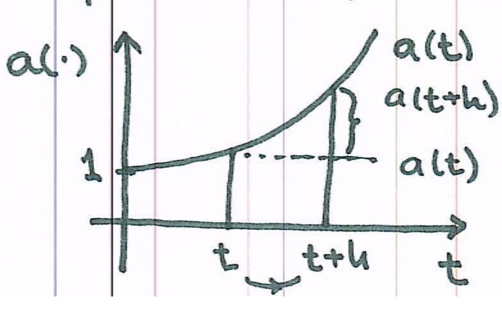


For every t, h, we define the REALIZED RETURN:

$$\boxed{R(t, t+h) := \ln\left(\frac{S(t+h)}{S(t)}\right)}$$

↑ a random variable

Consider an accumulation function in the compound interest case. Let r be the continuously compounded risk-free interest rate:



$$a(t)\, e^{r \cdot h} = a(t+h)$$

$$rh = \ln\left(\frac{a(t+h)}{a(t)}\right)$$

Recall: $\boxed{\sigma \ldots \text{volatility}}$

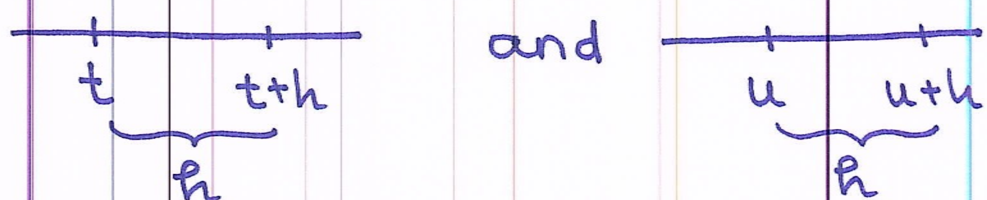
the standard deviation of the realized return over any time period of length one year

$\Rightarrow$ we should have:

$$\mathrm{Var}[R(0,1)] = \sigma^2 \quad \text{, i.e.,} \quad SD[R(0,1)] = \sigma$$

Interpretation of our modeling assumptions in continuous time

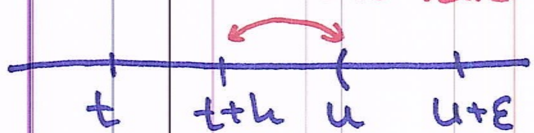
i.



We require that $R(t, t+h)$ and $R(u, u+h)$ be identically dist'd.

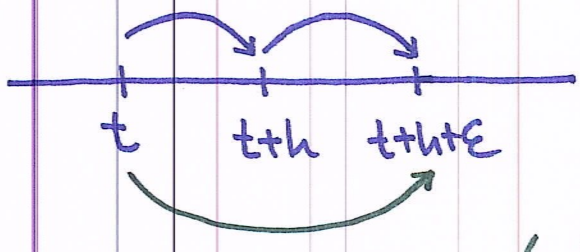
ii.

we can have them equal if we want



$R(t, t+h)$ and $R(u, u+\varepsilon)$ are required to be independent.

iii •



$$R(t, t+h+\varepsilon) = \ln\left(\frac{S(t+h+\varepsilon)}{S(t)}\right)$$

by def'n

$$= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)} \cdot \frac{S(t+h)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)}\right) + \ln\left(\frac{S(t+h)}{S(t)}\right)$$

$$= R(t, t+h) + R(t+h, t+h+\varepsilon)$$

Realized returns are ADDITIVE.

$\Rightarrow$ We decide to model our realized returns

$R(t, t+h)$ as normally distributed,

i.e.,

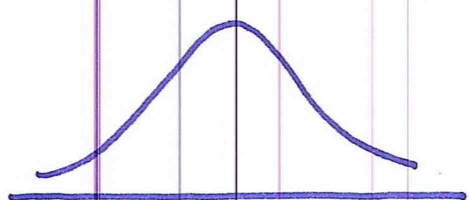
$$R(t, t+h) \sim N(\text{mean} = m \quad , \text{variance} = v^2)$$

The Normal Distribution.

Standard normal random variable:

$$Z \sim N(\text{mean}=0, \text{var}=1)$$

Its law is given by its probability density f'tion (pdf):



$$\left(\varphi(z)=\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, z\in\mathbb{R}$$

$$=: f_Z(z)$$

$\Rightarrow$ Its cumulative dist'n f'tion (cdf):

$$N(a) = \mathbb{P}[Z \leq a] = \int_{-\infty}^{a} f_Z(z)\,dz$$

... we find in the std normal tables or using the (Prometric) calculator

$$\boxed{N'(z) = f_Z(z)}$$