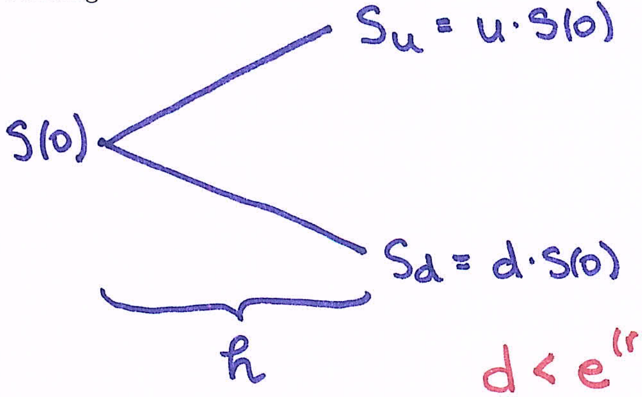
UNIVERSITY OF TEXAS AT AUSTIN

January 24th, 2020

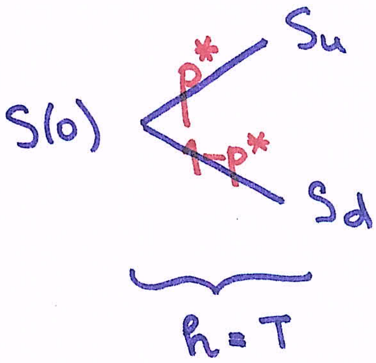
Binomial option pricing (review).

Problem 1.1. Let the continuously compounded risk-free interest rate be denoted by r. You are building a model for the price of a stock which pays dividends continuously with the dividend yield δ. Consider a binomial tree modeling the evolution of the stock price. Let the length of each period be h and let the up factor be denoted by u, and the down factor by d. What is the **no-arbitrage** condition for the binomial tree you are building?



$$d < e^{(r-\delta)h} < u$$

Problem 1.2. Set up the framework for pricing by replication in a one-period binomial tree! What is the *risk-neutral pricing* formula?



PAYOFF

$V_u = v(S_u)$

$V_d = v(S_d)$

e.g. for a K-strike call

$V_u = (S_u - K)_+$

$V_d = (S_d - K)_+$

p^* ... risk-neutral probab.

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

$$V(0) = e^{-rT} \cdot [p^* \cdot V_u + (1-p^*) V_d]$$

Generalizing: $\boxed{V(0) = e^{-rT} \mathbb{E}^*[V(T)]}$

$\delta = 0$ $S(0) = 100$

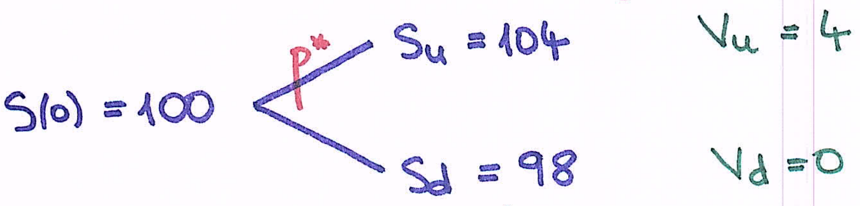
$h = \frac{1}{4}$

Problem 1.3. The current price of a certain non-dividend-paying stock is $100 per share. You are modeling the price of this stock at the end of a quarter year using a one-period binomial tree under the assumption that the stock price can either increase by 4%, or decrease by 2%. $\Rightarrow$ $u = 1.04,\ d = 0.98$

The continuously compounded risk-free interest rate is 3%. $r = 0.03$

What is the price of a three-month, at-the-money European call option on the above stock consistent with the above binomial tree? $K = 100$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.0075} - 0.98}{0.06} = 0.4588$$



②

$$V_c(0) = e^{-0.03(0.25)} \cdot 0.4588 \cdot 4 = 1.8215$$

Problem 1.4. Let the continuously compounded risk-free interest rate be equal to 0.04. $r = 0.04$

The current price of a continuous-dividend-paying stock is $80 and its dividend yield is 0.02. The stock's volatility is 0.25. You model the evolution of the stock price over the following half year using a two-period **forward** binomial tree. T n

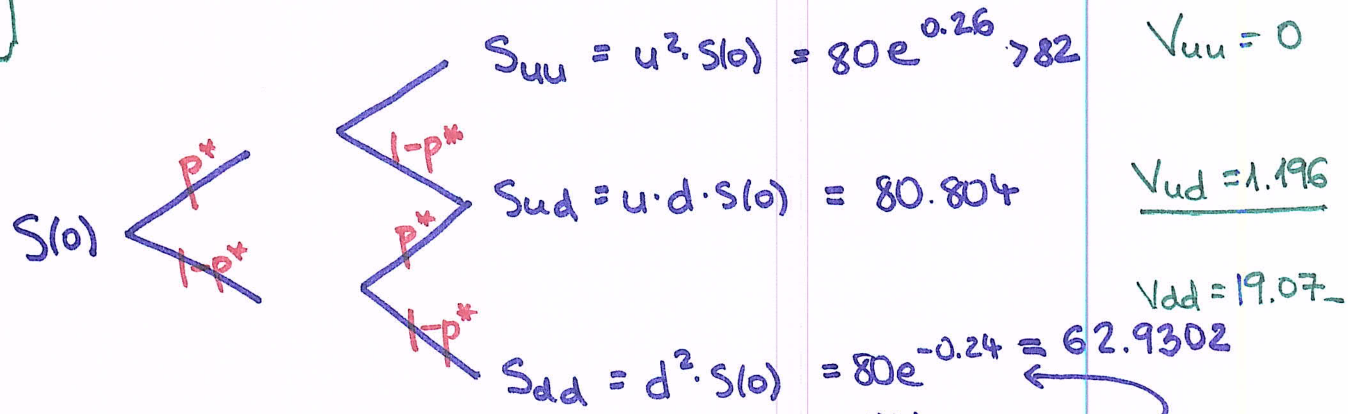
What is the price of a six-month, $82-strike European put option on the above stock consistent with the given binomial tree?

$h = 1/4$

In the FORWARD TREE: $p^* = \dfrac{1}{1+e^{\sigma\sqrt{h}}} = \dfrac{1}{1+e^{0.125}}$

$\Rightarrow p^* = 0.4688$

$\boxed{K = 82}$



$$\text{w/} \quad u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.04-0.02)(1/4) + 0.125} = e^{0.13}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{0.005 - 0.125} = e^{-0.12}$$

$$V_p(0) = e^{-0.04(1/2)} \cdot \left(2p^*(1-p^*) \cdot 1.196 + (1-p^*)^2 \cdot 19.07\right) = 5.85832$$

$S(0)$

Problem 1.5. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03. $\sigma = 0.2, \ \delta = 0.03$

The continuously compounded risk-free interest rate equals 0.06. $r = 0.06$

$K=95$ Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option. $T = 3/4$

↳ $n = 3 \Rightarrow h = 1/4$

(a) \$2.97
(b) \$3.06
(c) \$3.59
(d) \$3.70
(e) None of the above.

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.1}} = 0.475$$

↑ forward tree

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{0.03(0.25)+0.1} = 1.1135$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{0.03(0.25)-0.1} = 0.9116$$

$\boxed{K=95}$

$S_{ddd} = S(0)\cdot d^3 = 75.7553,$ $V_{d^3} \cong 19.25$ w/ $(1-p^*)^3$

$S_{ddu} = S(0)\cdot d^2\cdot u = 92.5335$ $V_{d^2u} \cong 2.5$ w/ $3p^*(1-p^*)^2$

the rest are out·o·money ⤴

$$V_p(0) = e^{-0.06(0.75)} \cdot \left(19.25(1-p^*)^3 + 2.5\cdot 3\cdot p^*(1-p^*)^2\right)$$

$= 3.5884 \Rightarrow$ (c) ■

③

Problem 1.6. Consider a non-dividend-paying stock whose current price is $100 per share. Its volatility is given to be 0.25. You model the evolution of the stock price over the following year using a two-period forward binomial tree.

$S(0)$

$\sigma = 0.25$

$T = 1$ $n = 2$

The continuously compounded risk-free interest rate is 0.04. $r = 0.04$

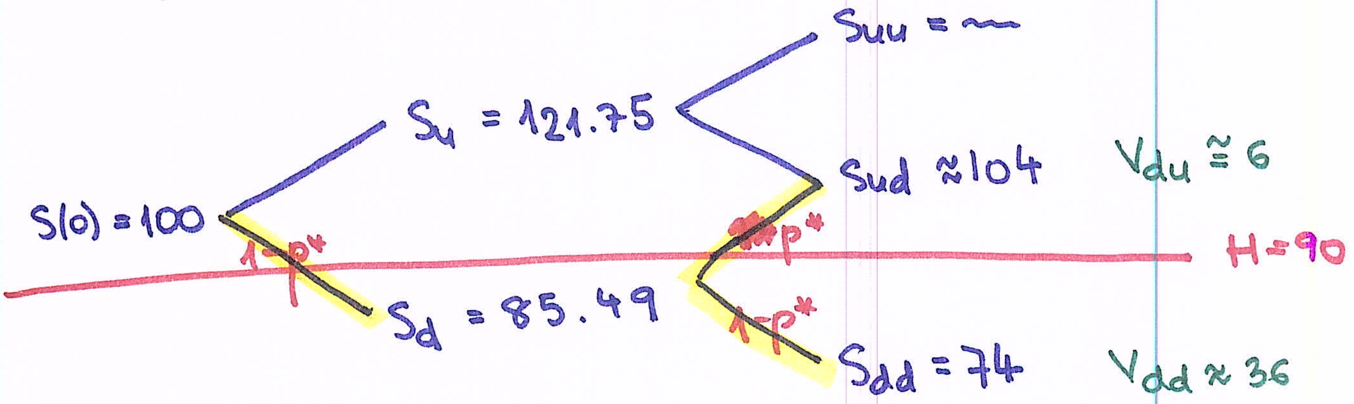
Consider a $110-strike, one-year **down-and-in** put option with a barrier of $90 on the above stock. What is the price of this option consistent with the above stock-price model?

- (a) About $10.23
- (b) About $11.55
- (c) About $11.78
- (d) About $11.90
- (e) None of the above.

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.25\cdot\frac{1}{\sqrt{2}}}} = \dots = 0.4559$$

$$u = e^{0.02+0.25\cdot\frac{1}{\sqrt{2}}} = 1.2175$$

$$d = e^{0.02-0.25\frac{1}{\sqrt{2}}} = 0.8549$$



$$V(0) = e^{-0.04(1)}\cdot\left((1-p^*)\cdot p^*\cdot 6 + (1-p^*)^2\cdot 36\right) \approx 11.91.$$