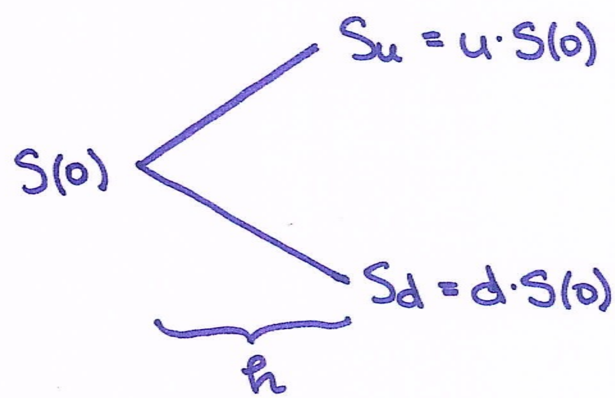
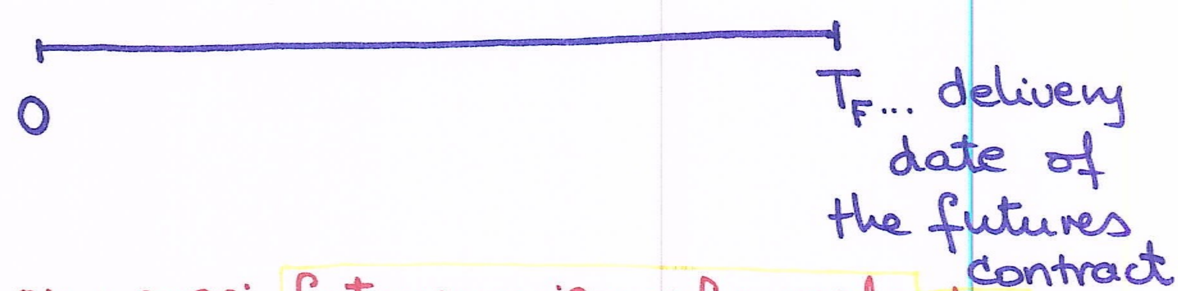
M339W: January 29th, 2020.

Binomial pricing: Futures options [cont'd]

Temporarily: focus on futures contracts on a continuous-dividend-paying market index w/ dividend yield δ.

STOCK-PRICE TREE





In our case: futures prices = forward prices

$\Rightarrow$ In general: $F_{t,T_F}(S) = S(t)\, e^{(r-\delta)(T_F - t)}$

$\Rightarrow$ In our futures/forward price tree:

- @ the ROOT node:

$$F_0 := S(0)\, e^{(r-\delta)\cdot T_F}$$

1.

• @ the up node:

$$F_u := S_u e^{(r-\delta)(T_F - h)}$$

$$= \underline{\underline{u}} \cdot \underbrace{S(0) e^{(r-\delta)\cdot T_F}}_{F_0} \cdot \underline{\underline{e^{-(r-\delta)\cdot h}}}$$

set:

$$\boxed{u_F := u \cdot e^{-(r-\delta)h}}$$

↑ up factor in the futures tree

$$\Rightarrow F_u = F_0 \cdot u_F$$

• @ the down node:

$$F_d := S_d \cdot e^{(r-\delta)(T_F - h)}$$

$$= d \cdot \underbrace{S(0)\, e^{(r-\delta)\cdot T}}_{F_0} \cdot e^{-(r-\delta)h}$$

$$\boxed{d_F := d \cdot e^{-(r-\delta)h}}$$

↑ down factor in the futures tree

$$\Rightarrow F_d = F_0 \cdot d_F$$

The Futures Price Tree:

$$\begin{tikzcd} & F_u = u_F \cdot F_0 \\ F_0 \arrow[ur] \arrow[dr] & \\ & F_d = d_F \cdot F_0 \end{tikzcd}$$

$$\underbrace{\qquad\qquad}_{h}$$

We generalize this same type of a tree to any underlying asset of the futures contract.

* The risk-neutral probability:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{\cancel{e^{(r-\delta)h}}\,1 - d_F \cdot \cancel{e^{(r-\delta)h}}}{u_F \cancel{e^{(r-\delta)h}} - d_F \cancel{e^{(r-\delta)h}}}$$

$$\Rightarrow \boxed{p^* = \frac{1-d_F}{u_F-d_F}}$$

Example. If you have a forward tree modeling futures prices, then use the analogy $\delta \leftrightarrow r$

$$\Rightarrow u_F = e^{(r-r)h + \sigma\sqrt{h}} = e^{\sigma\sqrt{h}}$$

$$d_F = e^{(r-r)h - \sigma\sqrt{h}} = e^{-\sigma\sqrt{h}}$$

$$\& \quad p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}$$

*46. You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is 6 months. $h = 1/2$

(ii) $u/d = 4/3$, where u is one plus the rate of gain on the futures price if it goes up, and d is one plus the rate of loss if it goes down. $\frac{u_F}{d_F} = \frac{4}{3}$

(iii) The risk-neutral probability of an up move is 1/3. $p^* = \frac{1}{3}$

(iv) The initial futures price is 80. $F_0 = 80$

(v) The continuously compounded risk-free interest rate is 5%. $r = 0.05$

$T = 1$ $K = 85$

Let C_I be the price of a 1-year 85-strike European call option on the futures contract, and C_{II} be the price of an otherwise identical American call option.

Determine $C_{II} - C_I$.

(A) 0

(B) 0.022

(C) 0.044

(D) 0.066

(E) 0.088

47. Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.

You are given the following information:

(i) The risk-free interest rate is constant.

(ii)

	Several months ago	Now
Stock price	\$40.00	\$50.00
Call option price	\$ 8.88	\$14.42
Put option price	\$ 1.63	\$ 0.26
Call option delta	0.794	

4.

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.

→: Given: $\frac{u_F}{d_F} = \frac{4}{3}$

$$p^* = \frac{1}{3} \underset{\uparrow}{=} \frac{1-d_F}{u_F-d_F} \cdot \frac{d_F}{d_F}$$

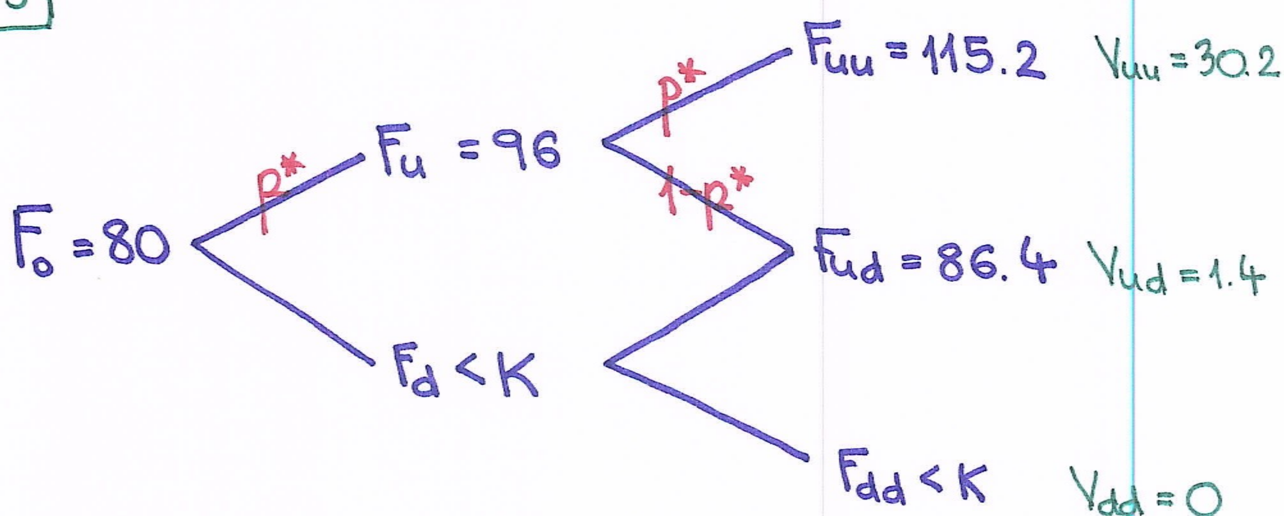
by our model

$$\Rightarrow \frac{1}{3} = \frac{\frac{1}{d_F}-1}{\underbrace{\frac{u_F}{d_F}-1}_{=\frac{1}{3}}} \Rightarrow \frac{1}{9} = \frac{1}{d_F} - 1$$

$$\Rightarrow d_F = 0.9 \quad \& \quad u_F = 1.2$$

Call

$\boxed{K=85}$



• up node: $V_u^E = CV_u = e^{-0.025}\left[\frac{1}{3}\cdot 30.2 + \frac{2}{3}\cdot 1.4\right]$

$$\left.\begin{array}{l} = 10.7284 \\ IE_u = 11 \end{array}\right\} \Rightarrow V_u^A = 11$$

• down node: out·o·money

$$\Rightarrow V_d^A = V_d^E$$

$$\Rightarrow C_{II} - C_I = e^{-0.025} \cdot \frac{1}{3} \cdot (11 - 10.7284) = 0.088$$

⇒ (E) ■

Subjective Probabilities

Individual investors / companies form a model of the probability dist'n of the time-T stock price $S(T)$.

Since any sensible model for $S(T)$ will be non-deterministic, the least we can consider to asset the quality of investment is $\mathbb{E}[S(T)]$.

Assume: Invest in a portfolio (among the admissible ones) which has the highest expected profit.

Note: Since we can invest @ the risk-free interest rate, we should demand $\mathbb{E}[\text{Profit of investment}] > 0$.