UNIVERSITY OF TEXAS AT AUSTIN

Lecture 1
Caps. Binomial Interest Rate Trees.

1.1. **Caps.** Building up on the reasoning for the introduction of swaps on interest rates, we introduce another contract: **the cap**. Caps consist of a set of **caplets** with different exercise dates. In turn, each caplet serves as a kind of a call option on the floating interest rate. It is not necessarily benefitial to break a cap into caplets before exploring its mechanism; it is not much more involved than a swap on an interest rate. It is more common and sensible to base the discussion on **effective interest rates** rather than the continuously compounded ones. To make sure you can follow the rest, make sure that you review the concepts of and the notation for spot and implied forward interest rates.

Consider a loan in the amount L and assume that the recepient of the loan is charged a floating interest rate with interest-only payments at the end of every year for the next n years. The time when the loan is issued is time-0. This means that the amount to be paid at time k by the borrower is $L \times R_{k-1}(k-1, k)$ for $k = 1, \ldots, n$, where $R_{k-1}(k-1, k)$ denotes the interest rate charged at time $k - 1$ for the loan taken at that time and to be repaid at time k. This is the *floating rate* we are aiming to insure ourselves against using a cap. Note that the floating rate is always

(1) known at time $k - 1$ for the next year, and
(2) only known at that time; in particular, this is **not** the implied forward rate calculated at time-0 based on the spot rates at that time

Let the "strike" interest rate be denoted by K_R. Then, the borrower gets the difference between the interest payment based on the floating interest rate and the interest payment based on K_R, provided that this difference is positive. Since the interest payment for the kth year is made at time k, this particular excess amount is

$$(L \times R_{k-1}(k-1, k) - L \times K_R)_+ = L(R_{k-1}(k-1, k) - K_R)_+.$$

However, since $R_{k-1}(k-1, k)$ is already known at time $k - 1$, it is also sensible to make the payments (if any) at that time. If this is the case, taking into account the time value of money, we have to discount the above amount using the prevailing (floating) rate. So, the amount to be paid at time $k - 1$ is

$$\frac{L}{1 + R_{k-1}(k-1, k)}(R_{k-1}(k-1, k) - K_R)_+.$$

The only exception is time-0, when no payment to the cap holder is made even if the current one-year spot rate exceeds K_R. The reason is that this payment would happen at time-0 and one would simply have to price the cap differently and have the same cash amount flow in both directons.

If the payments are not interest only, then one needs to modify the above set up to incorporate the outstanding loan balance as it changes over time. The implementation is analogous to the one conducted for swaps on interest rates.

The whole contract setting up the above set of payments is referred to as the **cap**. A single one of the above payments is called a **caplet**.

Example 1.1. Consider a 7.5%-interest-rate cap on a $100 three-year loan whose interest payments are made annually.

Say that at time-1, we see that the prevailing interest rate for a one-year loan equals 7.7%. This means that at that time, the following payment is made to the cap holder:

$$\frac{L}{1 + R_1(1,2)}(R_1(1,2) - K_R)_+ = \frac{100}{1 + 0.077}(0.077 - 0.075) = 0.1857.$$

The consequence on the loan-repayment is that the borrower, having been compensated for the difference by the cap, pays only $0.075 \times 100 = 7.5$ at time-2.

If the prevailing rate for a one-year loan at time-2 equals 6%, then this value lies below $K_R = 0.075$ and there is no payment from the cap. The borrower pays $0.06 \times 100 = 6$ as his/her final interest payment at time-3.

1.2. **Binomial interest rate trees.** Here is the notation and conventions we are going to use with binomial interest rate trees:

- h is the length of the binomial period; if it is not stated otherwise, we take that a period is 1 year, i.e., $h = 1$
- $r_{t_0}(t, T)$ is the **forward interest rate** at time t_0 for time t to time T
- $r_{t_0}(t, T; j)$ is the interest rate from t to T, where the rate is quoted at time $t_0 < t$ and the **state** (i.e., the **height** of the corresponding node in the binomial tree) is j.

Both of the above types of interest rates can be understood as either effective or comtinuously compounded – the two set ups are interchangeable and bear the same information. It will be specified in the problem whether the interest rates are used as effective or continuously compounded, but the effective interest rates are more natural in this setting. The index j stands for the height of the node in a tree which is not necessarily recombining. We cannot just say that j stands for the number of "down" movements (or "up" movements) in the tree. In fact, as Example 1.2 below shows, it is best not to think of a binomial interest tree as recombining even if it happens to recombine (as it will be in the case of the Black-Derman-Toy model). If the number of periods in the tree is manageable (say, up to three), we will resort to using the indices along the lines of u, d, uu, etc. This approach is more descriptive and makes for simpler notation as we have seen in stock-price trees.

Note that the risk-neutral probability is not stipulated above. As opposed to the stock-price trees, the risk-neutral probability is **not** calculated based on the interest rates in the trees. It has to be exhogenously prescribed. It is very common to denote it by p (note the absence of the asterisk!) and, moreover, set it to be equal to $1/2$.

Note: Study the binomial tree in Figure 24.2 in McDonald. Do not get confused: this is indeed a schematic of the **three-period** interest rate model - the 1-year rate is observable today; then, the 1-year rate can go up or down over the first year (thus generating different 2-year rates); again, there can be up or down movement during the second year (resulting in a set of possible 3-year rates).

Example 1.2. <u>MFE Exam, Spring 2007: Problem #9.</u>
You use a binomial interest rate model to evaluate a 7.5%-interest-rate cap on a $100 three-year loan. You are given

(i) The interest rates in the binomial tree are:

$$r_0 = 6\%, \quad r_u = 7.704\%, \quad r_{uu} = 9.892\%,$$
$$r_d = 4.673\%, \quad r_{ud} = r_{du} = 6\%,$$
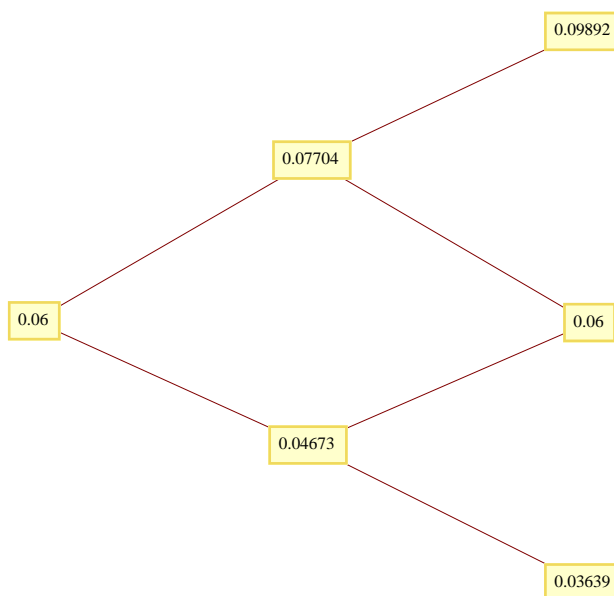$$r_{dd} = 3.639\%.$$

(ii) All interest rates are annual **effective** rates.
(iii) The risk-neutral probability that the annual effective interest rate moves up or down is $1/2$.
(iv) The loan-interest payments are made annually.

Using the binomial interest rate model, calculate the time-0 value, i.e., the price, of this interest rate cap.

Solution. Let us first display the interest rate tree.

The only two nodes at which the strike-interest-rate $K_R = 7.5\%$ lies below the realized interest rate are the "up" and the "up-up" node. So, these are the only two nodes at which a positive payment takes place. At the "up" node, the payment amount is

$$\frac{100}{1 + r_u}(r_u - K_R) = 0.1894.$$

At the "up-up" node, the payment amount is

$$\frac{100}{1 + r_{uu}}(r_{uu} - K_R) = 2.1767.$$

The present value of the payment from equation (1.1) equals

$$0.1894/(1 + r_0) = 0.1894/1.06.$$

The payment from equation (1.1) occurs with the probability of reaching the "up" node. This probability is given to be $1/2$. On the other hand, the payment from equation (1.1)

has the present value

$$\frac{2.1767}{(1 + r_0)(1 + r_u)} = \frac{2.1767}{(1.06)(1.07704)}.$$

This payment occurs with the probability of reaching the "up-up" node, i.e., with probability given to be $(1/2)^2 = 1/4$. So, the price of the cap equals

$$\frac{0.1894}{2 \times 1.06} + \frac{2.1767}{4 \times 1.06 \times 1.07704} = 0.57.$$

   Notes:
   (1) Take a moment to contemplate the different discounting factors associated with different trajectories in the (recombining) binomial tree above. In particular, look at the two different ways in which the "up-down" node can be reached and the two different discountings that would have had to take place had the interest rate r_{ud} exceeded K_R.
   (2) Take another look at the expression from equation (1.1). See if you can optimize your calculations so that you do not waste time punching numbers into the calculator.

1.3. **Zero-coupon bond prices.** At any time t_0, there is a set of both spot (for $t = t_0$) and implied (for $t > t_0$) forward zero-coupon bond prices: $P_{t_0}(t, T; j)$ where j still denotes the height of a particular node in the tree. The mechanics of the zero-coupon-bond-pricing procedure are the following: Each path in the binomial tree implies a realized discount factor. Hence, we can value a bond by considering separately each path the interest rate can take. Then, we compute the discount factors going backwards down each path. Finally, we find the expected present value of the bond revenue using risk-neutral probabilities assigned in the model. Effectively, if our tree contains the continuously compounded interest rates, to evaluate the price of the zero-coupon bond redeemed for $1, we perform a calculaton of the following form:

$$\mathbb{E}^* \left[e^{-\sum_{i=0}^{n} r_i h} \right]$$

where n is the number of time-steps, r_i are the time-i rates from the binomial tree and $\mathbb{E}^*$ is the expectation with respect to the risk-neutral measure (as generated by the specified risk-neutral probabilities of the interest rate moving "up" in a single period denoted by p). In the case that the interest rates are effective, we behave analogously.

   You might be tempted to use an "average" interest rate in stead of the above procedure. This is an **incorrect** approach. You might get "close-enough" numbers for interest-rate models exhibiting a "reasonably-low" variance.

   Uncertainty causes bond yields to be lower than the expected average interest rate. This discrepancy increases with volatility (morally speaking – the variance) of interest rates. Therefore, we **cannot** price a bond by using the expected interest rate.

Example 1.3. Let us use the tree from Example 1.2 to find the price of a zero-coupon bond redeemable for $1,000 in three years. We begin by evaluating the discounting coefficients for every path in the tree.

up-up:

$$(1 + r_0)^{-1}(1 + r_u)^{-1}(1 + r_{uu})^{-1} = (1.06 \times 1.07704 \times 1.09892)^{-1} = 0.79707$$

up-down:

$$(1 + r_0)^{-1}(1 + r_u)^{-1}(1 + r_{ud})^{-1} = (1.06 \times 1.07704 \times 1.06)^{-1} = 0.826336$$

down-up:

$$(1 + r_0)^{-1}(1 + r_d)^{-1}(1 + r_{ud})^{-1} = (1.06 \times 1.04673 \times 1.06)^{-1} = 0.850264$$

down-down:

$$(1 + r_0)^{-1}(1 + r_d)^{-1}(1 + r_{dd})^{-1} = (1.06 \times 1.04673 \times 1.03639)^{-1} = 0.869633$$

Due to $p = 1/2$, all of the above factors are equally likely and occur with probability $1/4$. So, the bond price equals

$$1,000 \times \frac{0.79707 + 0.826336 + 0.850264 + 0.869633}{4} = 835.83.$$

Note: It is possible to implement the above procedure by moving backwards throught the tree one step at a time. The by-product are the set of bond prices for different maturities and at different times. The computational complexity is not much different from the above method. Try to convince yourseles of this fact.

Now that it's evident that the bond prices can be evaluated by moving through the above tree, we can consider options on bonds and value them using binomial interest-rate trees. Consider a European option on the bond. Let the payoff function of the option be denoted by $v : \mathbb{R}_+ \to \mathbb{R}_+$. Let the option's expiration date be denoted by T_o (assume this is the end of a period in the interest-rate tree). The set up is sensible only if the maturity date for the bond T satisfies $T_o \leq T$. Let the value of the option at time t (the end of a time period in the interest-rate tree) and at height j, be denoted by $v_{t,j}$. Then, the value at this option on its exercise date T_o is a function of the bond price at that time, i.e., $v_{T_o,j} = v(P_t(t, T; j))$. From the payoffs we move backwards through the non-recombining tree using the recursion

$$v_{t-h,j} = P_{t-h}(t - h, t; j)[pv_{t,2j+1} + (1 - p)v_{t,2j}]$$

where p stands for the specified risk-neutral probability.

Example 1.4. Call on a bond.
Consider a two-period binomial interest-rate tree with risk-neutral probability set at $1/2$. Assume that the root effective annual interest rate equals $r_0 = 0.05$. From then on, the interest rates change according to an up factor equal to $u = 1.2$ and the down factor $d = 0.8$. So, the tree has the following form:

0.06

0.05

0.04
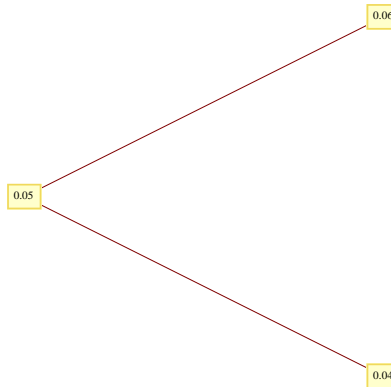
Consider a zero-coupon bond redeemable for \$1 in two years. At the "up" node, the value of this bond equals $P_u := P_1(1, 2; u) = 1/(1 + r_u) = 0.943396$. At the "down" node, the value of this bond equals $P_d := P_1(1, 2; d) = 1/(1 + r_d) = 0.961538$. Setting the risk-neutral probability at $p = 1/2$. At the root node, the bond is worth

$$P(0) := P_0(0, 2; 0) = \frac{1}{2} \times \frac{1}{1 + r_0}(P_u + P_d) = \frac{1}{2} \times \frac{1}{1.05} \times (0.943396 + 0.961538) = 0.907112$$

The bond-price tree looks like this:

0.943396

0.907112

0.961538
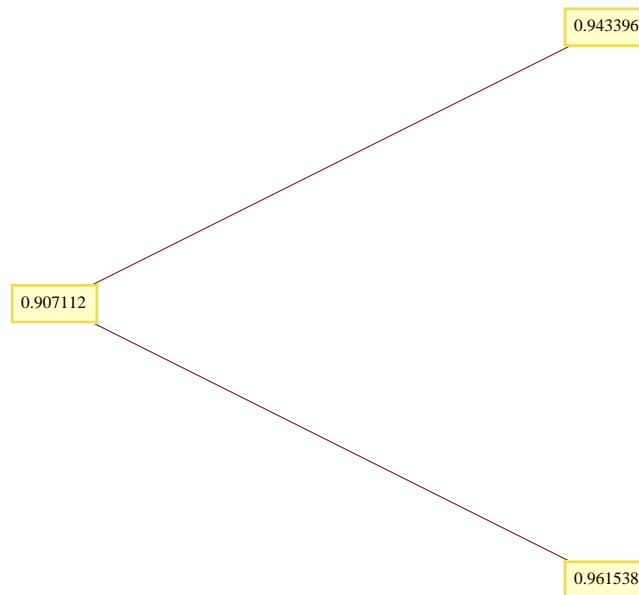
Next, we want to price a call option on the above bond with exercise date at time $T_o = 1$ and with strike $K = 0.95$. This option is not exercised at the "up" node since the strike exceeds P_u; so, payoff of this option at the "up" node is $v_u = 0$. At the "down" node, the payoff is $v_d = 0.961538 - 0.95 = 0.011538$. So, the price of this call is

$$v_0 = \frac{1}{1.05} \times \frac{1}{2} \times 0.011538 = 0.0054942.$$

Example 1.5. Getting p using existing bond prices.
Assume that the binomial interest-rate tree is populated with the following effective annual interest rates:

$$r_0 = 0.04, r_u = 0.045, r_d = 0.035.$$

We observe that a zero-coupon bond redeemable in two-years for $100 is priced at $P(0) = \$92.5$ using the above interest-rate model. What is the risk-neutral probability p stipulated by the model?
Solution. Using the same method as in Example 1.4, we get

$$P_u = 100/1.045 = 95.6938, \quad P_d = 100/1.035 = 96.6184.$$

Our models do not allow arbitrage, so the risk-neutral probability p must satisfy
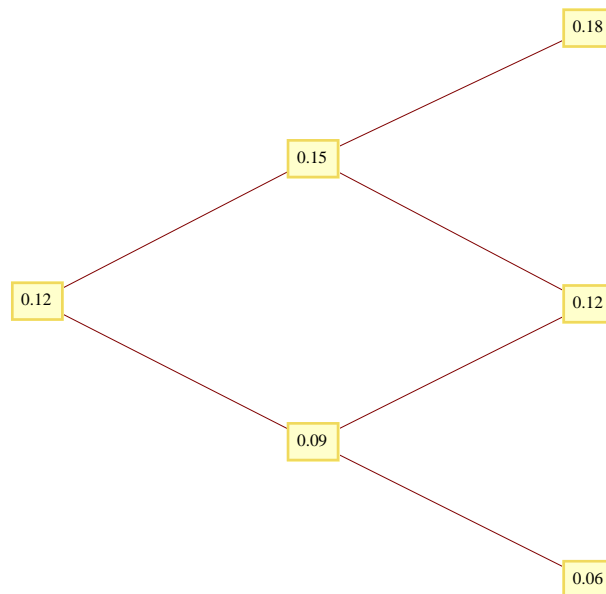
$$92.5 = P(0) = \frac{1}{1 + r_0} \times [p \cdot P_u + (1 - p) \cdot P_d] = \frac{1}{1.04} \times [95.6938p + 96.6184(1 - p)].$$

Hence,

$$p = \frac{96.6184 - 1.04 \times 92.5}{96.6184 - 95.6938} \approx 0.4525.$$

Example 1.6. MFE Exam, Spring 2009: Problem #5.
You are given the following three-period interest rate tree. Each period is one year. The risk-neutral probability of each up-move is $p = 70\%$. The interest rates are **continuously compounded** rates on the annual basis.

Consider a European put option that expires in 2 years, giving you the right to sell a one-year zero-coupon bond for $0.90. This zero-coupon bond pays $1 at maturity. Determine the price of the put option.

Solution. There are altogether four paths through the above interest tree. The bond prices at the leaves are:

up-up:

$$P_{uu} = e^{-r_{uu}} = 0.83527$$

up-down:

$$P_{ud} = e^{-r_{ud}} = 0.88692$$

down-up:

$$P_{du} = e^{-r_{ud}} = 0.88692$$

down-down:

$$P_{dd} = e^{-r_{dd}} = 0.941765$$

The put payoffs at the above nodes are

$$v_{uu} = (K - P_{uu})_+ = 0.0647298,$$
$$v_{ud} = v_{du} = (K - P_{ud})_+ = 0.01308,$$
$$v_{dd} = (K - P_{dd})_+ = 0.$$

The put-price is, finally,

$$V_P(0) = e^{-r_0} \left[pe^{-r_u}(pv_{uu} + (1-p)v_{ud}) + (1-p)e^{-r_d}(pv_{du} + (1-p)v_{dd}) \right] = 0.0285359.$$

Because the tree can be used at any node to value zero-coupon bonds of any maturity, the tree also generates implied forward interest rates of all maturities, as well as volatilities of implied forward rates. In fact, we can equivalently specify a binomial interest rate tree in terms of any of the following:

1. Interest rates (either effective or continuously compounded)
2. Zero-coupon bond prices
3. Volatilities of implied forward interest rates