M339W: March 11th, 2020.

T=1

✱ **41.** Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

(i) The time-0 stock price is 45. $S(0)=45$

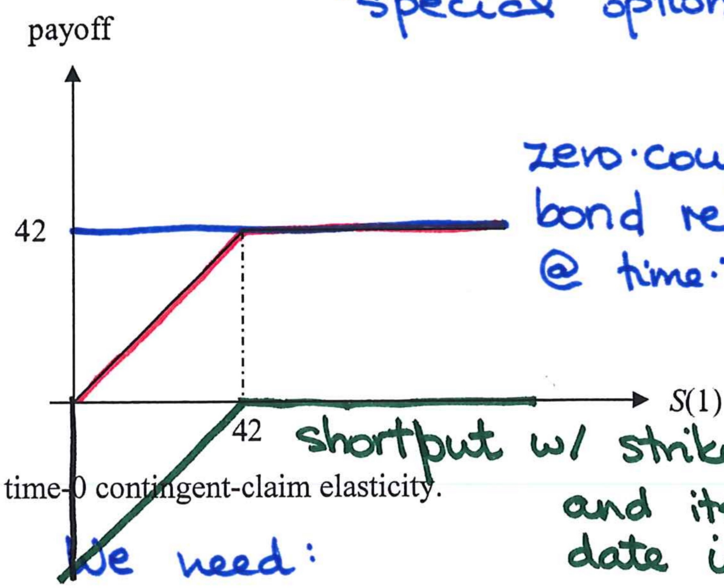
(ii) The stock's volatility is 25%. $\sigma=0.25$

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\delta=0.03$

(iv) The continuously compounded risk-free interest rate is 7%. $r=0.07$

(v) The time-1 payoff of the contingent claim is as follows:

"special option"



zero-coupon bond redeemable @ time·T for \$42

short put w/ strike @ 42 and its exercise date is T=1

Calculate the time-0 contingent-claim elasticity.

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

We need:

$$\Omega_{SO}(S(0),0)=?$$

By def'n: $\Omega_{SO}(S(0),0)=\dfrac{\Delta_{SO}(S(0),0)\cdot S(0)}{v_{SO}(S(0),0)}$

We designed a replicating portfolio for our special option:

- bond (as above)

and

- short put (as above)

1.

Thus, for any arguments (s,t):

$$v_{SO}(s,t) = v_B(s,t) - v_P(s,t)$$
$$= 42\, e^{-r(T-t)} - v_P(s,t)$$

$\frac{\partial}{\partial s}\Big\backslash$

$$\Delta_{SO}(s,t) = 0 - \Delta_P(s,t) = -\left(-e^{-\delta(T-t)} \cdot N(-d_1(s,t))\right)$$

$$\boxed{\Delta_{SO}(s,t) = e^{-\delta(T-t)} N(-d_1(s,t))}$$

With the B·S put price:

$$v_{SO}(s,t) = 42e^{-r(T-t)} - \left(42\, e^{-r(T-t)} \cdot N(-d_2(s,t)) - s e^{-\delta(T-t)} N(-d_1(s,t))\right)$$

$$v_{SO}(s,t) = 42e^{-r(T-t)} \cdot N(d_2(s,t)) + s e^{-\delta(T-t)} N(-d_1(s,t))$$

$\Rightarrow$ <u>At time 0</u>:

$$d_1(S(0)=45, 0) = \frac{1}{0.25\sqrt{1}}\left[\ln\left(\frac{45}{42}\right) + \left(0.07 - 0.03 + \frac{(0.25)^2}{2}\right)\cdot 1\right]$$

$$= 0.56$$

$$\Rightarrow d_2(S(0)=45,0) = 0.56 - 0.25 = 0.31$$

$$\Rightarrow N(-d_1(S(0),0)) = N(-0.56) = 1 - N(0.56)$$
$$= 1 - 0.7123 = 0.2877$$

$$N(+d_2(S(0),0)) = N(0.31) = 0.6217$$

$$\Rightarrow v_{SO}(S(0),0) = 42e^{-0.07} \cdot (0.6217)$$
$$+ 45e^{-0.03}(0.2877) \approx 36.91$$

$$\Delta_{SO}(S(0),0) = e^{-0.03}(.2877) = 0.2797$$

$\Rightarrow$ Finally:

$$\Omega_{SO}(S(0),0) = \frac{0.2797 \cdot 45}{36.91} \approx 0.34 \Rightarrow (C) \blacksquare$$

Q: What's the volatility of the contingent claim?

$\rightarrow$: $$\sigma_{SO} = \sigma \cdot |\Omega_{SO}|$$

at time 0 : $\sigma_{SO}(S(0),0) = 0.25 \cdot 0.34$

$= \ldots$ $\blacksquare$

The Gamma

Γ... the second order sensitivity of the price w/ respect to the changes in the price of the underlying, i.e.,

$$\Gamma(s,t) := \frac{\partial^2}{\partial s^2} v(s)$$

In the Black-Scholes model, the Γ of the call option:

$$\Gamma_c(s,t) = \frac{\partial^2}{\partial s^2} v_c(s,t) =$$

$$= \frac{\partial}{\partial s}\Big(\underbrace{\frac{\partial}{\partial s} v_c(s,t)}\Big)$$

$$= \frac{\partial}{\partial s}\left(\Delta_c(s,t)\right)$$

$$= \frac{\partial}{\partial s}\left(e^{-\delta(T-t)} \cdot N(d_1(s,t))\right)$$

$$= e^{-\delta(T-t)} \underbrace{\frac{\partial}{\partial s} N(d_1(s,t))}_{\parallel \text{ chain rule}}$$

$$\underbrace{N'(d_1(s,t))}_{\varphi(d_1(s,t))} \cdot \frac{\partial}{\partial s} d_1(s,t)$$

PHI

$$d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(s) - \ln(K) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)\right]$$

$$\Rightarrow \frac{\partial}{\partial s} d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{s}$$

$$\Rightarrow \Gamma_c(s,t) = e^{-\delta(T-t)} \cdot \varphi(d_1(s,t)) \cdot \frac{1}{s \cdot \sigma\sqrt{T-t}}$$

Q: What's the put gamma?

By put-call parity:

$$v_c(s,t) - v_p(s,t) = s\underbrace{e^{-\delta(T-t)}}_{} - Ke^{-r(T-t)}$$

$\frac{\partial}{\partial s}\Big\backslash$

$$\Delta_c(s,t) - \Delta_p(s,t) = e^{-\delta(T-t)}$$

$\frac{\partial}{\partial s}\Big\backslash$

$$\boxed{\Gamma_c(s,t) = \Gamma_p(s,t)}$$