Implied Volatility.

W: March 29th, 2019.

* We can observe put/call prices in the market.
* Assuming the Black·Scholes model

$\Rightarrow$ we have formulae for put/call prices:

$$\underbrace{v(s, t, r, \delta}_{\text{Assumed to be given/observed}}, \sigma)$$

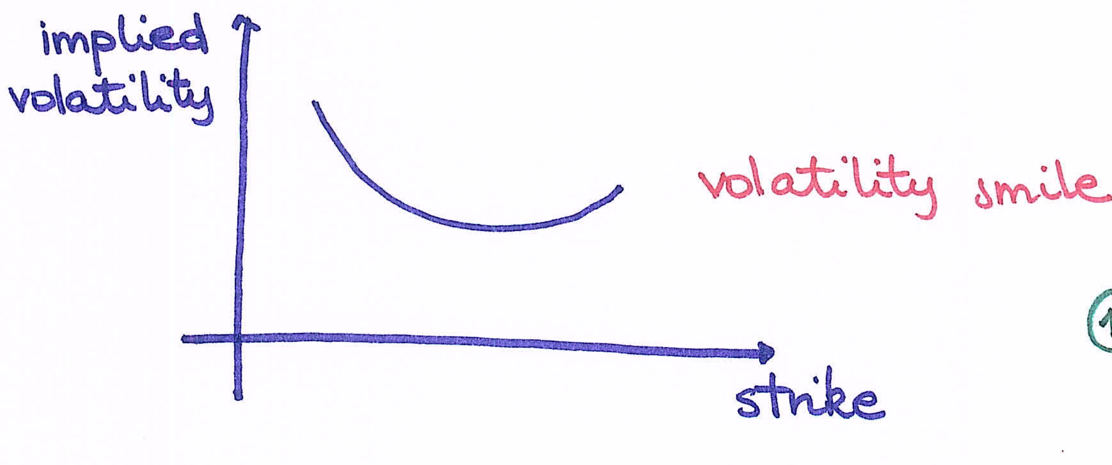
Look @ the pricing formula as a function of σ.

$\downarrow$

Invert the Black·Scholes price and get the σ which is then called the implied volatility

Theoretically: If all of the above assumptions are true, the observed call prices for varying strikes K should give us the same implied volatility σ.

Practically:



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17. Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.

You are given:

* (i) The ratio of the put option price to the stock price is less than 5%.

(ii) Delta of the put option is –0.4364.

(iii) The continuously compounded risk-free interest rate is 1.2%.

Determine the stock's volatility.

(A) 12%

(B) 14%

(C) 16%

(D) 18%

(E) 20%

(ii) $\Delta_p(S(0),0) = -0.4364$

$= -e^{-\delta(T)} \cdot N(-d_1(S(0),0))$

$\underset{\uparrow}{=} -N(-d_1(S(0),0))$

nodividends

$\Rightarrow N(-d_1(S(0),0)) = 0.4364$

$\Rightarrow N(d_1(S(0),0)) = 1-0.4364 = 0.5636$

$\Rightarrow d_1(S(0),0) = N^{-1}(0.5636) = 0.16$

$$\| $$

$$\frac{1}{\sigma \underbrace{\cancel{\sqrt{T}}}_{T=1}}\left[\underbrace{\cancel{\ln\left(\frac{S(0)}{K}\right)}}_{\text{at-the-money}} + \left(r - \cancel{\delta} + \frac{\sigma^2}{2}\right) \cdot \overset{T=1}{\cancel{T}}\right]$$

$\Rightarrow \frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right) = 0.16$

(iii)

$$\Rightarrow \quad 0.012 + \frac{\sigma^2}{2} = 0.16 \cdot \sigma \qquad / \cdot 2$$

$$\Rightarrow \quad \sigma^2 - 0.32\sigma + 0.024 = 0$$

$$\Rightarrow \quad \sigma_1 = 0.12 \quad \text{and} \quad \sigma_2 = 0.20$$

$$\text{(i)} \Rightarrow \frac{V_p(0)}{S(0)} < 0.05$$

$$\frac{\cancel{K}e^{-r \cdot T} \cdot N(-d_2(S(0),0)) - \cancel{S(0)} \cdot N(-d_1(S(0),0))}{\cancel{S(0)}} < 0.05$$

at the money

$$e^{-r \cdot T} \cdot N(-d_2(S(0),0)) - \underbrace{N(-d_1(S(0),0))}_{0.4364 \text{ (given in (ii))}} < 0.05$$

$$e^{-r \cdot T} \cdot N(-d_2(S(0),0)) < 0.4864$$

$$N(-d_2(S(0),0)) < \underbrace{e^{0.012} \cdot 0.4864}$$

__Note__: $d_2 = d_1 - \sigma\sqrt{T}$,

i.e., $d_2 = 0.16 - \sigma$

$$N(-d_2(S(0),0)) = N(\sigma - 0.16) \overset{?}{<} e^{0.012} \cdot 0.4864$$

Test : $\sigma_1 = 0.12$ and $\sigma_2 = 0.20$

Or just think : Choose $\sigma_1 = 0.12$ ∎

Example. Consider an at-the-money option w/ $r = \delta$, or a non-dividend-paying stock w/ strike $K = S(0)e^{rT}$, or any choice of given values such that

$$\underbrace{F^P_{0,T}(S) = PV_{0,T}(K)}$$

If this is the case:

$$d_1(S(0),0) = \frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{F^P_{0,T}(S)}{PV_{0,T}(K)}\right)} + \frac{\sigma^2 \cdot T}{2}\right] = \frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

For the call price, we get:

$$v_C(S(0),0) = \underline{S(0)e^{-\delta \cdot T}} \cdot N(d_1(S(0),0)) - \underline{Ke^{-r \cdot T}} \cdot N(d_2(S(0),0))$$

$$= F^P_{0,T}(S)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

$$= F^P_{0,T}(S)\left(2 \cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

If we're given the price of the call option, we can (using the std normal tables) invert the price function using the above formula. ■

Delta · Gamma · Theta Approximation

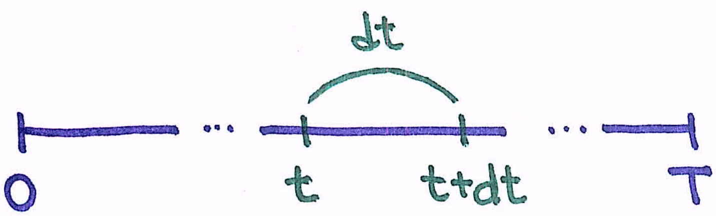
In our market model, we have:

- **risk-free** asset, i.e., borrowing/lending money @ the ccrfir r
- **risky** asset, i.e., say, a stock w/ price denoted by $\underbrace{S(t), t \geq 0}_{\text{stochastic process}}$ per share

$\Rightarrow$ Also, derivative securities on S are available.

Assume the Black · Scholes model on S!

Focus on portfolios in the above market model: We can consider the value function of this portfolio: $\boxed{v(s,t)}$



$$S(t) \qquad S(t+dt) = S(t)e^{(r-\delta-\frac{\sigma^2}{2})dt + \sigma\sqrt{dt}\cdot Z}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$s \qquad\qquad s+ds$$

$$v(s,t) \longrightarrow v(s+ds, t+dt)$$

Taylor-like Expansion

$$\begin{aligned} v(s+ds, t+dt) \cong\ & v(s,t) \\ & + \underbrace{\frac{\partial}{\partial s} v(s,t)}_{=:\Delta(s,t)} ds \\ & + \frac{1}{2} \underbrace{\frac{\partial^2}{\partial s^2} v(s,t)}_{=:\Gamma(s,t)} (ds)^2 \\ & + \underbrace{\frac{\partial}{\partial t} v(s,t)}_{=:\Theta(s,t)} dt \end{aligned}$$

Delta-Gamma-Theta Approximation